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Foundations of time-symmetric physics. 1. Time-symmetric relativity and relativistic quantum mechanics

Zahid Zakir[1]

Abstract

Time-symmetric theory (TST) is formulated as a theoretical foundation of time-symmetric physics (TSP), classical, relativistic and quantum. TST is based on the widely used in particle physics fact that the description of positive energy antiparticles moving forward in space and time is equivalent to the description of negative energy particles moving backward in space and time (Zisman 1940, Stückelberg 1941, Feynman 1949). In TST, this fundamental fact is generalized by formulating the equivalence principle of particle physics and applying it to all physical phenomena in the rest frames of negative energy particles. As a result, the group of transformations TST includes such inertial frames also that move backward in space and time with coordinates related to the ordinary ones by 4-inversion. This led to the formulation of time-symmetric relativity theory and corresponding relativistic quantum mechanics. In the latter, the probabilities remain positive, but the bilinear forms of the wave functions are associated with probability currents, the signs of which depend on the direction of movement along the coordinate axes, including the time axis. In TST, therefore, the Klein-Gordon equation is a consistent equation for probability amplitudes, and the fermion theory contains the necessary corrections, which make it physically more consistent, unchanging its main observational consequences. Applications of TST to quantum fields will be considered in the second article.

Keywords: antiparticles, vacuum energy, scalar particle, fermions, Klein-Gordon equation, Dirac equation, probabilistic interpretation

Content

[1] *Center for Theoretical Physics and Astrophysics, Tashkent, Uzbekistan*; zzakir@qgph.org, ORCID

1. Introduction

The formalism of relativistic quantum theory leads to negative energy particles, the physical interpretation of which has long time remained controversial. In the 1940s the Zisman-Stückelberg-Feynman (ZSF) interpretation resolved the main of these contradictions [1,2,3]. This interpretation uses the formal analogy or the equivalence between the description of positive energy antiparticles moving forward in space and time, and negative energy particles moving backward in space and time. Unlike the hole theory, applicable only to fermions and leading to divergent energy and charge of the vacuum, the ZSF interpretation is applicable to all particles and does not have problems with the ground state energy and charge. The ZSF interpretation is widely used in particle physics as a convenient way of covariant description of antiparticles in terms of particles and reveals in crossing symmetry [4,5].

In the previous articles [6], a method of time-symmetric quantization (TSQ) of relativistic fields was formulated, in which, firstly, quantum field theory becomes compatible with ZSF interpretation, and secondly, the problem with divergences in the energy and charge of the vacuum of free fields disappears. If ZSF interpretation naturally provides in particle physics the covariance of the description and the vanishing of the energy and charge of the vacuum, then it is also necessary to reconcile with this interpretation its theoretical foundations - the theory of relativity and quantum mechanics. This is the aim of the present and next articles - to formulate the foundations of *time-symmetric theory* (TST), including time-symmetric formulations of the relativity theory, relativistic quantum mechanics and quantum field theory.

In TST, the fundamental fact expressed in ZSF interpretation is generalized and *the equivalence principle of particle physics* is formulating on its basis, which is applied to the rest frames of negative energy particles also. Thus, in TST, not only antiparticles, but their rest frames also are described as moving backward in time, with the time axis directed into the past of ordinary time. Therefore, the Poincaré group should be expanded to such inertial frames, the coordinates of which are related with ordinary ones by spacetime inversion (4-inversion). TST appear as a natural formalism for description of systems with antiparticles and *CPT*-symmetry.

It will be shown also that TST eliminates the contradictions and misunderstandings in the previous formulation of relativistic quantum theory, which became elements of the standard paradigm for historical reasons. In particular, the bilinear forms of wave functions, which were associated with probabilities, are in fact associated with probability currents, the sign of which depends on the direction of motion of the particle in space or in one of two directions of time.

In the case of fermions, the physical picture does not change and there appear only some technical complications associated with spinors. TST introduces small corrections making the theory of fermions physically more consistent without changing its observational consequences.

In Section 2, ZSF interpretation and the equivalence principle of particle physics are described and then TST based on this principle is formulated. In Section 3, the time-symmetric relativistic quantum mechanics of a scalar particle, and in Section 4 of a fermionic particle are formulated. Time-symmetric quantum field theory will be considered in the next article. More detailed discussion of TST and its consequences will be presented in the book [7].

2. Time-symmetric relativity theory

2.1. ZSF interpretation of negative energy particles

The equivalence between the description of antipositive energy particles moving forward in space and time and the description of negative energy particles moving backward in space and time was discovered by G. Zisman in 1940 [1] and later in 1941 by E. Stückelberg [2]. In 1949 R. Feynman, citing Stückelberg, used this equivalence in the propagator approach to relativistic quantum mechanics and developed the diagram technique on its basis [3]. This approach then successfully competed with quantum field theory and forms the basis of the standard course of relativistic quantum mechanics (see [4,5]).

Stückelberg and Feynman considered this way of describing antiparticles pragmatically, as a convenient heuristic method for the covariant description of quantum processes with antiparticles, especially useful in diagram technique.

Zisman not only discovered this equivalence first, but also gave a more fundamental justification for it, starting from the concepts of relativistic theory even before their application to quantum theory. He showed that the proper time of all particles always increases along their trajectories, while the time of the ordinary reference frame where the particle is moving can decrease if the trajectory goes backward in this time, which is the case for negative energy particles [1] (see also [8]).

Here this approach will be discussed briefly, and in the next section it will be generalized by introducing rest frames of negative energy particles and formulating a new physical principle for them.

In relativistic theory t, time measured by a clock in an inertial frame K, and the proper time of a particle τ, measured by a clock in its rest frame K', are related as:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\mathbf{r}^2, \tag{1}$$

$$d\tau = \pm dt\sqrt{1-\mathbf{v}^2/c^2}, \tag{2}$$

where $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{r})$ are the coordinates of the particle in K.

Usually the sign in front of the root in (2) was taken to be positive, considering that the direction of the course of τ always coincides with the direction of the course of t. This choice corresponded to the choice of the positive energy particles, while the negative sign, as it was shown by Zisman [1,8], corresponds to the choice of the negative energy particles, which can be used to describe antiparticles.

This approach was the basis for ZSF interpretation, but in TST, which will be formulated in the next section, it is supplemented with a fundamentally important element. Namely, the fact that in fact the sign in (1) depends on the combination of the directions of evolution of the particle and the frame of reference where the motion of the particle is described. In particular, for particles moving backward in ordinary time, this sign will be positive in the frame of reference that also moves backward in ordinary time.

For 4-velocity u^μ is defined by the following relationships:

$$u^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{c\,d\tau} = \frac{dx^\mu}{cdt}\frac{dt}{d\tau} = (u_0, \mathbf{u}) = (1, \mathbf{v}/c)\frac{dt}{d\tau}, \tag{3}$$

$$u_0 = \frac{dt}{d\tau} = \pm\frac{1}{\sqrt{1-\mathbf{v}^2/c^2}}. \tag{4}$$

Therefore, for the 4-momentum of a particle p^μ, taking into account $u^\mu u_\mu = 1$, we have:

$$p^\mu = mcu^\mu = m\frac{dx^\mu}{d\tau} = (p^0, \mathbf{p}), \quad p^\mu p_\mu = m^2c^2, \tag{5}$$

where m is the rest mass. For the energy $E = cp_0$ the relations (3)-(5) give:

$$E = mc^2\frac{dt}{d\tau} = \pm\frac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}} = \pm c\sqrt{\mathbf{p}^2 + m^2c^2}. \tag{6}$$

Thus, a negative sign in (2) leads to a negative sign of the energy in (6).

If the energy of a particle can, in principle, have two signs, then questions arise about whether such states exist and, if so, what their physical interpretation is. The answers turned out to be nontrivial: such states do not exist in nature, but they can be introduced as a way to describe positive energy antiparticles. Such a description is not only a beautiful and convenient heuristic method widely used in particle physics, but also turns out to be a natural language for the formalism of relativistic theory. For a covariant description of antiparticles, it is necessary to extend the formulation of the relativistic theory to negative energy particles also.

To do this, let us consider, from a slightly different point of view, the question of what a change in the sign of energy in (6) means. As a guide, let us consider the factor that determines the signs of the spatial components of the particle momentum. In the frame K the 3-momentum vector $\mathbf{p}$ can be represented as the sum of its projections onto the coordinate axes with unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$: $\mathbf{p} = p^1\mathbf{e}_1 + p^2\mathbf{e}_2 + p^3\mathbf{e}_3$. Here its coordinates are defined as:

$$\mathbf{p}\cdot\mathbf{e}^1 = p^1 = m\frac{dx^1}{d\tau} = mv^1\frac{dt}{d\tau},... \tag{7}$$

and their sign depends on the product of the signs of velocity $v^1 = dx^1/dt$ and $dt/d\tau$.

For positive energy particles $E > 0$ always $dt/d\tau > 0$ and the sign of p^1 coincides with the sign v^1. Therefore, at moving along the axis x^1 of the frame K with a constant velocity $v^1 > 0$, the momentum is positive $p^1 > 0$ and with increasing τ along the trajectory $\Delta\tau > 0$ the time and spatial coordinates also increase $\Delta t > 0$ and $\Delta x^1 > 0$. If later the particle's velocity changes sign and becomes negative $v^1 < 0$, then the momentum will also become negative $p^1 < 0$. Then with growth τ along the trajectory $\Delta\tau > 0$ only the time coordinate increases $\Delta t > 0$, while the spatial coordinate decreases: $\Delta x^1 < 0$. Thus, when the direction of motion of a particle changes from direct to reverse, the corresponding coordinate decreases and the momentum changes sign, and vice versa, if the momentum is negative, then the particle moves in the direction opposite to the given axis. In all cases, τ grows along the trajectory $\Delta\tau > 0$.

Returning to the clarification of the factor that determines the sign of energy, we see that the situation here is similar, but now we are talking about evolution along the time axis and a change in its direction with a change in the corresponding component of momentum p^0, proportional to energy E.

If at $\tau_{(1)} = 0$ a free positive energy particle $E > 0$ rested at the point $\mathbf{r} = 0$ of the frame K and $\Delta\mathbf{r} = 0$, then its world line is parallel to the axis t and $dt/d\tau > 0$ along it. During of further evolution τ and t increase: $\Delta\tau > 0$, $\Delta t > 0$. If the particle energy was negative $E < 0$, then its world line is directed opposite to the axis t and, according (6), will be $dt/d\tau < 0$, i.e. with growth τ along the trajectory t will decrease: $\Delta\tau > 0$, $\Delta t < 0$.

Thus, changes in the signs of the 4-momentum components p^1 are p^0 explained in the same way and are associated with changes in the directions of particle motion relative to the corresponding axes of the coordinate system in space of events.

It can be further shown that diagrams in which the world lines of a particle going up turn and begin to go down, and vice versa, going down, turn and begin to go up, are similar to the processes of annihilation and creation of particle-antiparticle pairs. This illustrates the equivalence between the motion of positive energy antiparticles moving forward in time and the motion of negative energy particles moving backward in time. This formal equivalence is used in ZSF interpretation, on the basis of which the propagator approach was created.

2.2. Time-symmetric evolution and the equivalence principle of particle physics

Classical, relativistic and quantum mechanics were created to describe a world of positive energy particles moving forward in time in frames of reference whose bases also consist of the same particles and also move forward in time. Antiparticles, which are also of positive energy, are described as the same particles, differing only by their charges, i.e. as charge-conjugate ones. But such a description of antiparticles is non-covariant, and the corresponding non-covariant diagram technique with world lines of particles and antiparticles is very cumbersome.

The covariant diagram technique is very compact, since it uses world lines only of particles moving forward and backward in time and therefore ZSF interpretation is used here either explicitly or implicitly. For example, one single-loop diagram for a vertex corresponds to 6 diagrams of the non-covariant diagram technique with an explicit indication of the world lines of antiparticles and ordering in time of the three vertices.

The formalisms of all three types of mechanics also allowed for a second possibility - descriptions of negative energy particles moving backward in time. But in this case it is necessary to use ZSF interpretation and, as will be shown below, a transition to more general physics containing fundamentally new elements is also required. The fact that such a transition doubles the number of degrees of freedom is not something fundamentally new, since this also occurs in the non-covariant formulation with only positive energies.

But the really new elements of the theory are the following. At first, the rest frames of negative energy particles, where their proper times are measured, are comoving these particles and therefore also are moving backward in time. In this case, the movement of the reference frame backward in time means that its basis, scales and clocks are built from antiparticles described in terms of particles going back in time. And secondly, this circumstance requires an extension of the formalism of mechanics, since it is necessary to include a group of transformations between inertial frames, the axis of the time coordinate in which, where the course of proper time is measured, is directed into the past of ordinary time. And thirdly, there are transitions between the coordinates of frames with opposite time axes. All this introduces fundamentally new elements both into formalism and into the physical aspects of theories.

In particular, the creation of a particle-antiparticle pair is depicted as the evolution into the past of a particle of negative energy, which at the creation point of the pair changes the direction of evolution and begins to move forward in time. Thus, the particle is firstly rested in a frame with a reverse time axis, and then with a normal axis. When a pair annihilates, on the contrary, at first the particle is rested in a frame with the ordinary time axis, and then with the reverse axis.

Let us consider the description of a particle flow with a 3-velocity field $\mathbf{v} = d\mathbf{r}/dt$, taking into account the presence of particles with inverse evolution in time. The total number of particles N is

$$N = \int n(x)\, d^3x = \int n\, dV, \tag{8}$$

where $n(x)$ is the particle number density. Invariance N, as well as $dN = n\,dV$, the number of particles in the volume element dV, allows us to introduce a 4-vector of flux density j^μ, multiplied by an invariant 4-volume element d^4x (see [9]):

$$dN \cdot dx^\mu = n\frac{dx^\mu}{dt} dt dV = \frac{1}{c} j^\mu d^4x, \tag{9}$$

$$j^\mu = \frac{dx^\mu}{dt} n = (\pm cn, \mathbf{v}n) = (j^0, \mathbf{j}). \tag{10}$$

Here $j^0 = \pm cn$, $\mathbf{j} = \mathbf{v}n$ are the temporal and spatial components of j^μ. The magnitude of the time component of the 4-velocity $dx^0 / dt = \pm c$, although remained equal to the velocity of light c, but it has two signs for two types of particles evolving in different directions of the time axis. Integral of j^μ over the hypersurface S_μ

$$p_N = \int j^\mu dS_\mu \tag{11}$$

gives the total flow through this hypersurface, i.e. the number of particles crossing it per unit time. For a hypersurface $t = const$ with a normal $\mathbf{e}_0$, directed along the time axis, we get:

$$p_N = \int j^0 dS_0 = \int j^0 dV = \pm c \int n\, dV, \tag{12}$$

i.e. two particle flows in opposite directions of the time axis.

Now, instead of a flow of many particles, consider the flow of probability in an ensemble of systems, where each contains one fluctuating particle. The probability of a particle's position in space w is also invariant, and its value in a volume element dV is equal to $dw = \rho dV$, where $\rho(x)$ is the probability density. Determination of total probability

$$w = \int \rho\, dV, \quad \rho = \frac{dw}{dV} \tag{13}$$

allows us to enter a 4-vector of probability flux density j^μ:

$$dw \cdot dx^\mu = \rho \frac{dx^\mu}{dt} dt dV = \frac{1}{c} j^\mu d^4 x, \tag{14}$$

$$j^\mu = \frac{dx^\mu}{dt} \rho = (\pm c\rho, \mathbf{v}\rho) = (j^0_\pm, \mathbf{j}). \tag{15}$$

Here $j^0_\pm = \pm c\rho$, $\mathbf{j} = \mathbf{v}\rho$ are the temporal and spatial components of the probability flux density, and $dx^0 / dt = \pm c$ has two signs indicating the directions of motion along the time axis. The integral of this flow over the hypersurface S_μ gives the total probability current through this hypersurface and in the particular case of the hypersurface $t = const$ we obtain:

$$p_{w\pm} = \int j^0 dS_0 = \int j^0_\pm dV = \pm c \int \rho\, dV, \tag{16}$$

i.e. two probability currents for two particles moving in two directions of the time axis.

The principle of relativity, certainly, is also valid for systems with antiparticles, and therefore physical phenomena with negative energy particles in frames of reference moving backward in time must proceed in the same way as physical phenomena with positive energy particles in frames of reference moving forward in time. Such a generalization of the principle of relativity, as will be discussed in the next section, requires expanding the group of transformations to a new class of frames of reference.

Here we will consider another aspect of this situation, the question of how physical phenomena occur with negative energy particles in frames of reference moving forward in time. In the particular case of particle motion, the answer to this question is given by ZSF interpretation, which is in fact an additional postulate of the theory. Since this fundamental fact of particle physics is in any case included in the theory in the form of a postulate, it makes sense to introduce it in the most general form, making it applicable for both classes of frames of reference and for all physical systems with both signs of energy.

Therefore, as a generalization of ZSF interpretation, we formulate *the principle of equivalence of particle physics*, which consists in the statement that in the frames of reference moving forward in time, physical phenomena with negative energy objects moving backward in time are equivalent to physical phenomena with charge-conjugate them positive energy objects moving forward in time.

This principle, together with the generalization of the principle of relativity, means a transition to more general physics, TSP, and to TST as its theoretical basis. In the following sections, the main statements of TST will be considered, at first in relativity theory and then in quantum theory.

2.3. Time-symmetric transformations of frames of reference

Let's consider frames of reference and coordinate systems in 4-dimentional spacetime, taking into account the possibility of backward in time evolution.

Let's start with the more familiar case of ordinary 3-space with a coordinate system (x_+^1, x_+^2, x_+^3), where two opposite flows of particles intersect the coordinate plane (x_+^1, x_+^2) with a normal $\mathbf{e}_+^3$ along the axis x_+^3, the first flow from bottom to top, and the second from top to bottom. In this coordinate system, the first flow with $\mathbf{j}_+^{(+)} = \rho\mathbf{v}_+^{(+)}$ moves in the same direction as the normal $\mathbf{e}_+^3$, and therefore it is positive: $\mathbf{j}_+^{(+)} \cdot \mathbf{e}_+^3 > 0$, and the second flow c $\mathbf{j}_+^{(-)} = \rho\mathbf{v}_+^{(-)}$, directed in the opposite direction, is negative: $\mathbf{j}_+^{(-)} \cdot \mathbf{e}_+^3 < 0$.

Let us also introduce a second coordinate system (x_-^1, x_-^2, x_-^3) with axis inversion $x_-^i = -x_+^i$ and a plane (x_-^1, x_-^2) with a normal $\mathbf{e}_-^3$ along the axis x_-^3:

$$x_-^i = a_j^i x_+^j = -x_+^i, \quad a_j^i = -\delta_j^i. \tag{17}$$

In this coordinate system, the first flow moves in the opposite direction to the normal $\mathbf{e}_-^3$ and is therefore negative: $\mathbf{j}_-^{(+)} \cdot \mathbf{e}_-^3 < 0$, while the second flow moves in the direction of this normal and therefore it is positive: $\mathbf{j}_-^{(-)} \cdot \mathbf{e}_-^3 > 0$.

Let in the spacetime with coordinates $x_+^\mu = (x_+^0, x_+^1, x_+^2, x_+^3)$ there is a two-dimensional surface in space (x_+^1, x_+^2) at $x_+^3 = const$. The picture of the time evolution of particles on this plane in the 3-space of events (x_+^0, x_+^1, x_+^2) will be similar to two opposite flows in the usual 3-space. Positive energy particles cross the hypersurface $x_+^0 = const$ from bottom to top, and negative energy particles cross it from top to bottom. In this coordinate system, the first flow along the time axis moves in the same direction as the normal $\mathbf{e}_+^0$, which is directed along the time axis x_+^0, and therefore this flow is positive: $\mathbf{j}_+^{0(+)} \cdot \mathbf{e}_+^0 > 0$, and the second flow, directed opposite to the time axis, is negative: $\mathbf{j}_+^{0(-)} \cdot \mathbf{e}_+^0 < 0$.

Let us introduce a second coordinate system in spacetime $x_-^\mu = (x_-^0, x_-^1, x_-^2, x_-^3)$, where both the spatial and time axes are subject to inversion. The two coordinate systems x_+^μ and x_-^μ are related by the 4-inversion of spacetime coordinates PT:

$$x_-^\mu = a_\nu^\mu x_+^\nu = -x_+^\mu, \quad a_\nu^\mu = -\delta_\nu^\mu. \tag{18}$$

In the second coordinate system, the first flow moves in the opposite direction to the normal $\mathbf{e}_-^0$ and is therefore negative: $\mathbf{j}_-^{0(+)} \cdot \mathbf{e}_-^0 < 0$, while the second flow moves in the direction of this normal and therefore it is positive: $\mathbf{j}_-^{0(+)} \cdot \mathbf{e}_-^0 > 0$.

The rest frame of a positive energy particle is K_+, the basis of which consists of positive energy particles. Let in the initial hypersurface $x_+^0 = 0$ the particle be at the beginning of the 3-coordinate system $x_+^1 = x_+^2 = x_+^3 = 0$ in coordinates x_+^μ with the time axis $x_+^0 = ct_+$. At subsequent moments of time, the particle and its rest frame K_+ move up along the time axis of the initial coordinate system x_+^μ. The world lines of the particle and the origin of coordinates K_+ thereby form a real time axis, coinciding with the finite part of the infinite mathematical axis x_+^0 of the initial coordinate system.

Let at the initial moment a particle of negative energy also was at rest relative to the first particle. Its rest frame becomes an inertial frame K_-, the basis of which consists of negative energy particles, and the coordinate system x_-^μ is related by the coordinate system x_+^μ by the 4-inversion transformation *RT* (18). Along the time axis $x_-^0 = ct_-$, directed backward to the axis x_+^0, the proper time of the given particle is counted. At subsequent moments of time, this particle and the beginning of the system K_- will move down along the initial axis x_+^μ and up in the initial axis x_-^0. Thus, the world lines of the particle and the origin of coordinates K_- will form a real time axis in spacetime, coinciding with a finite part of the infinite mathematical axis x_+^0 in the lower light cone of K_+ and the axis x_-^0 in the upper cone of K_-.

Thus, particles of two energy signs, which at the initial moment $t_+ = t_- = 0$ were near the point $\mathbf{r} = 0$, evolve in time in the opposite directions. In a frame with a time axis x_+^0 a positive energy particle and its rest frame K_+ will go up into the future $x_+^0 > 0$, and a particle with negative energy and its rest frame K_- will go down into the past $x_+^0 < 0$.

Thus, the evolution of particles and their rest frames occurs symmetrically in time. Accordingly, coordinate transformations in these two classes of frames of reference will be symmetrical also. The usual Lorentz transformations $a_\nu^\mu(\mathbf{v}_+)$ relate the coordinates of the inertial frames K_+ and K'_+ with relative velocity $\mathbf{v}_+$:

$$x_+^{\mu}{}' = a_{\nu++}^{\mu} x_+^{\nu}. \tag{19}$$

Formally, exactly the same transformations will be added to them, relating the coordinates of the inertial frames K_- and K'_- with relative velocity $\mathbf{v}_-$:

$$x_-^{\mu}{}' = a_{\nu--}^{\mu} x_-^{\nu}. \tag{20}$$

The matrix $a_{\nu--}^{\mu}$ is the same as $a_{\nu++}^{\mu}$, but with $\mathbf{v}_-$ instead of $\mathbf{v}_+$.

When describing a particle of one energy sign in the coordinates of a frame of another energy sign, it becomes added the 4-inversion transformation *RT* (18):

$$x_+^{\mu}{}' = a_{\nu+-}^{\mu} x_-^{\nu}, \qquad x_-^{\mu}{}' = a_{\nu-+}^{\mu} x_+^{\nu}. \tag{21}$$

In the simplest case of describing in K_+ a particle of negative energy in the frame moving backward in space and time, its coordinates from the rest frame K_- are transformed by $a^{\mu}_{\nu+-}$, including the inversion of the time axis. This means that in Lorentz transformations $\pm\sqrt{1-\mathbf{v}^2/c^2}$ a negative sign is taken from two signs of the root and, in particular, the transformation of the time coordinate takes the form:

$$x^{0}_{+}{}' = -\frac{x^0_- - \mathbf{x}_- \cdot \mathbf{v}_- / c}{\sqrt{1-\mathbf{v}^2/c^2}} \tag{22}$$

Transformations of the extended Poincaré group, which also includes translations of the origin of coordinates in both types of coordinate systems, take the form:

$$x^{\mu}_{+}{}' = a^{\mu}_{\nu++} x^{\nu}_{+} + b^{\mu}_{+}, \tag{23}$$

$$x^{\mu}_{-}{}' = a^{\mu}_{\nu--} x^{\nu}_{-} + b^{\nu}_{-}. \tag{24}$$

$$x^{\mu}_{+}{}' = a^{\mu}_{\nu+-} x^{\nu}_{-} + b^{\nu}_{+}. \tag{25}$$

$$x^{\mu}_{-}{}' = a^{\mu}_{\nu-+} x^{\nu}_{+} + b^{\nu}_{-}. \tag{26}$$

A more detailed description of the properties of transformations of inertial frames moving in two directions of time will be given in subsequent articles and in the book [7].

Notice that the 4-volume element d^4x will remain invariant under these transformations when an even number of coordinate axes, including the time axis, are inverted.

2.4. Time-symmetric mechanics of a particle

In K_+ the action function for free positive energy particles and antiparticles is the same:

$$S = -mc\int_{s_0}^{s_1} ds = -mc^2\int_{\tau_0}^{\tau_1} d\tau = -mc^2\int_{t(\tau_0)}^{t(\tau_1)} dt\cdot\frac{d\tau}{dt} = -mc^2\int_{t_0}^{t_1} dt\sqrt{1-\mathbf{v}_+^2/c^2}, \quad t_1 > t_0, \tag{27}$$

and

$$S = \int_{t_0}^{t_1} dt\, L(\mathbf{v}_+^2) = \int_{t_0}^{t_1} dt\, L_+, \quad L_+ = -mc^2\sqrt{1-\mathbf{v}_+^2/c^2}, \quad t_1 > t_0. \tag{28}$$

Here $\tau_1 > \tau_0$, $t_1 = t(\tau_1)$, $t_0 = t(\tau_0)$ and $d\tau/dt = \sqrt{1-\mathbf{v}_+^2/c^2}$.

In K_+ the proper time of negative energy particles, still increases along their trajectory $\tau_1 > \tau_0$, but now these particles, according to (6), move in the direction of decreasing time coordinate t and $d\tau/dt = -\sqrt{1-\mathbf{v}_-^2/c^2}$. The action function for such a particle takes the form:

$$S = -mc\int_{s_0}^{s_1} ds = -mc^2\int_{\tau_0}^{\tau_1} d\tau = -mc^2\int_{t(\tau_0)}^{t(\tau_1)} dt\cdot\frac{d\tau}{dt} = mc^2\int_{t_0}^{t_1} dt\sqrt{1-\mathbf{v}_-^2/c^2}, \quad t_1 < t_0. \tag{29}$$

and

$$S = -\int_{t_0}^{t_1} dt\, L(\mathbf{v}_-^2) = \int_{t_0}^{t_1} dt\, L_-, \quad L_- = mc^2 \sqrt{1 - \mathbf{v}_-^2 / c^2}, \quad t_1 < t_0. \tag{30}$$

Here the Lagrange function L_- differs from L_+ by the sign.

The interaction can mix the states of both signs of energy and therefore it is convenient to write all time integrals in K_+ with the same limits $t_1 > t_0$. To do this, (30) we take the same moments of time as the limits of integration in (28). This results in an integral with initially rearranged limits:

$$S = -\int_{t_1}^{t_0} dt\, L(\mathbf{v}_-^2) = \int_{t_1}^{t_0} dt\, L_-, \quad t_1 > t_0, \tag{31}$$

and upon subsequent manual rearrangement of the limits, the sign of the integral changes:

$$S = \int_{t_0}^{t_1} dt\, L(\mathbf{v}_-^2) = -\int_{t_0}^{t_1} dt\, L_-, \quad t_1 > t_0. \tag{32}$$

As a result, the action function of two energy signs particles takes a compact form with the same limits of time integration:

$$S = \int_{t_0}^{t_1} dt\, [L(\mathbf{v}_+^2) + L(\mathbf{v}_-^2)] = \int_{t_0}^{t_1} dt\, (L_+ - L_-), \qquad t_1 > t_0. \tag{33}$$

In the general case of a system with two signs of energy, the generalized coordinates $q_\pm = (q_\pm^1, ..., q_\pm^n)$ and their generalized velocities $\dot{q}_\pm$ determine the Lagrange function $L_\pm = \pm L(q_\pm, \dot{q}_\pm)$, where $L(q_+, \dot{q}_+)$ is for a positive energy particle and $L(q_-, \dot{q}_-)$ is the same Lagrange function, but with the variables q_+ and $\dot{q}_+$ replaced by q_- and $\dot{q}_-$. The action then takes the form:

$$S = \int_{t_0}^{t_1} dt\, (L_+ - L_-) = \int_{t_0}^{t_1} dt\, [L(q_+, \dot{q}_+) + L(q_-, \dot{q}_-)]. \tag{34}$$

In the canonical formulation, the generalized momenta $p_\pm$, as well as the Hamilton functions $H_+ = H(q_+, \dot{q}_+)$, $H_- = -H(q_-, \dot{q}_-)$ are defined as:

$$p_\pm = \pm \frac{\partial L_\pm}{\partial \dot{q}_\pm} = \frac{\partial L(q_\pm, \dot{q}_\pm)}{\partial \dot{q}_\pm}, \tag{35}$$

$$H_\pm = p_\pm \dot{q}_\pm - L_\pm, \quad H_\pm = \pm H(q_\pm, p_\pm). \tag{36}$$

The Hamilton-Jacobi equations following from (34)-(36) have the form:

$$\pm \frac{\partial S}{\partial t} + H_\pm = 0, \quad \frac{\partial S}{\partial t} + H(q_\pm, p_\pm) = 0. \tag{37}$$

For extended objects and fields, the densities of the Lagrange function L and the Hamilton function H should be introduced:

$$\begin{aligned} S &= \int d^4x [\mathsf{L}_+(q_+, \dot{q}_+) + \mathsf{L}_-(q_-, \dot{q}_-)] = \\ &= \int d^4x [p_+ \dot{q}_+ + p_- \dot{q}_- - \mathsf{H}_+(q_+, p_+) - \mathsf{H}_-(q_-, p_-)]. \end{aligned} \tag{38}$$

Since integration in (38) is carried out over the 4-volume, which is invariant for an even number of inversions of the coordinate axes in the spacetime, L is also invariant under such transformations of frames in both light cones. The corresponding equations of motion are:

$$\dot{q}_{\pm} = \frac{\partial \mathsf{H}_{\pm}}{\partial p_{\pm}}, \quad \dot{p}_{\pm} = \mp \frac{\partial \mathsf{H}_{\pm}}{\partial q_{\pm}}. \tag{39}$$

In the presence of interactions, mixing two states, transition to K_+ with the same limits of time integrals $t_1 > t_0$ gives for (34)-(39):

$$\begin{aligned} S &= \int_{t_0}^{t_1} dt\, L(q_+, \dot{q}_+; q_-, \dot{q}_-) = \\ &= \int_{t_0}^{t_1} dt\, [p_+\dot{q}_+ + p_-\dot{q}_- - H(q_+, p_+; q_-, p_-)]. \end{aligned} \tag{40}$$

The Hamilton-Jacobi equations following from (34)-(36) have the form:

$$\frac{\partial S}{\partial t} + H(q_+, p_+; q_-, p_-) = 0. \tag{41}$$

For extended objects and fields, we have:

$$\begin{aligned} S &= \int d^4x\, \mathsf{L}(q_+, \dot{q}_+; q_-, \dot{q}_-) = \\ &= \int d^4x\, [p_+\dot{q}_+ + p_-\dot{q}_- - \mathsf{H}(q_+, p_+; q_-, p_-)]. \end{aligned} \tag{42}$$

3. Time-symmetric relativistic quantum mechanics of scalar particle

3.1. Relativistic equation for probability amplitude

In the relativistic theory, for the complex wave function $\psi(x)$ of a free scalar particle with 4-momentum $p^{\mu} = (p^0, \mathbf{p})$, according to (5), we obtain the equation:

$$p^{\mu} p_{\mu} \psi = m^2 c^2 \psi, \tag{43}$$

$$p_{0\pm} = \pm E_p / c, \quad E_p = c\sqrt{\mathbf{p}^2 + m^2 c^2}. \tag{44}$$

Inserting into (43) the coordinate representation $p^{\mu} = i\hbar \partial^{\mu}$ gives the Klein-Gordon (KG) equations for the complex probability amplitude or the wave function:

$$(\partial^{\mu}\partial_{\mu} + \kappa^2)\psi = 0, \quad (\partial^{\mu}\partial_{\mu} + \kappa^2)\psi^* = 0, \tag{45}$$

where $\kappa = mc/\hbar$. The solutions to these equations ψ and ψ^* describe the initial and final states correspondingly. Each of them contains states for two signs of energy $\psi_{\pm}$ and $\psi_{\pm}^*$, in particular, for the initial state there are two Hilbert spaces with basis vectors:

$$\psi_+ = a_+ \exp\left[-\frac{i}{\hbar}(E_p t - \mathbf{px})\right], \quad \psi_- = a_- \exp\left[\frac{i}{\hbar}(E_p t - \mathbf{px})\right], \tag{46}$$

where $a_{\pm}$ are normalization constants. In ZSF interpretation, particles with $p_{0-} < 0$ move backward in time and describe antiparticles with $p_{0+} > 0$ moving forward in time.

In the non-relativistic theory, the probability of a state w in a region of space is the integral over the volume of this region V over the probability density $\rho(x)$, and ρ is directly related to the square of the modulus of the non-relativistic limit of the wave function $\breve{\psi}$:

$$w = \int \rho \, dV = \int \breve{\psi}^* \breve{\psi} \, dV. \tag{47}$$

In relativistic theory, the wave functions ψ, ψ^*, satisfying the wave equations (45), are also related to the probability density ρ, but differently than in the non-relativistic theory, and there are two reasons for this.

On the one hand, in Eq. (45) the complex wave function ψ, describing the state of a spinless particle, must be a scalar (or pseudoscalar) function, as well as square of its modulus $\psi^*\psi$.

On the other hand, in relativistic theory, the expression $\pm\rho c = j^0$ becomes the 4-component of the vector $j^\mu = (j^0, \mathbf{j})$, probability flux density from (15), and the definition (47) is replaced by:

$$p_{w\pm} = \int j^0_\pm dV. \tag{48}$$

Now $p_{w\pm}$ is the probability current along the time axis and can have both signs depending on the sign $j^0_\pm$, the projection of 4-vector of flux density onto the time axis.

As a result, firstly, direct identification of the scalar $\psi^*\psi$ with ρ, which underlies nonrelativistic quantum mechanics, is now impossible and an expression for $j^0_\pm$ through the wave functions ψ and ψ^* must be clarified.

Secondly, $p_{w\pm}$ in (48) expresses not the probability of the state, but the integral of j^μ, the projection of the 4-vector of probability current onto the normal to the hypersurface $t = const$. If the projections of the components of $\mathbf{j}$ onto the spatial axes give the probability current densities along these axes, then the component $j^0_\pm$, the projection onto the time axis, gives the probability current densities along the axis t.

Thirdly, since $j^0_\pm$ is a projection onto the axis t, then for the probability current of positive energy particles going in the direction of this axis, the projection is positive definite $j^0_+ \geq 0$, and for the probability current of negative energy particles going backward to the axis, this projection is negative definite $j^0_- \leq 0$. Thus, in the relativistic quantum theory, the probability densities remain be positive-definite $\rho \geq 0$, and the different sign of the probability currents for particles of different energy signs expresses the oppositeness of their motion along the time axis.

To express $j^0_\pm$ through ψ and ψ^* an analogue of the flux density vector is constructed as is done in the non-relativistic theory - the first of the equations in (45) should be multiplied from the left by ψ^*, and the second by ψ and then the second expression should be subtracted from the first:

$$\psi^* \partial_\mu \partial^\mu \psi - \psi \partial_\mu \partial^\mu \psi^* = \partial_\mu (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) = 0. \tag{49}$$

Then substitution of coefficients gives an expression for the conserved vector j^μ:

$$j^{\mu}=\frac{i\hbar}{2m}(\psi^{*}\partial^{\mu}\psi-\psi\,\partial^{\mu}\psi^{*})=(j_{0},\mathbf{j}),\qquad \partial_{\mu}j^{\mu}=0, \tag{50}$$

the temporal and spatial components of which have the form:

$$j^{0}=\frac{i\hbar}{2mc}(\psi^{*}\partial_{t}\psi-\psi\,\partial_{t}\psi^{*}), \tag{51}$$

$$\mathbf{j}=-\frac{i\hbar}{2m}(\psi^{*}\nabla\psi-\psi\nabla\psi^{*}). \tag{52}$$

Thus, in relativistic quantum mechanics, the wave functions ψ are ψ^{*} not expressed directly through the probability density ρ, but are expressed through the projection of the 4-component of the probability flux density $j_{\pm}^{0}$.

For free particles with wave functions (46), the spatial component of flux is equal to $\mathbf{j}=\mathbf{p}/m$. Likewise, the flux component $j_{0\pm}$ also has different signs depending on the signs of the energy. For wave functions (46), when normalized to $2E_{p}$ particles per unit volume $V=1$, probability currents for states with different energy signs are given by the integrals:

$$p_{w}=\int j_{\pm}^{0}\,dV=\pm\frac{E_{p}}{mc}. \tag{53}$$

This expression is always positive for positive energy particles and always negative for negative energy particles.

Thus, in contrast to the previous interpretations, TST correctly takes into account the physical meaning $j_{\pm}^{0}$ as the probability flux density along the time axis, which has two signs for particles of two energy signs.

Thus, TST excludes from the theory one of the absurd statements of the previous formulation of relativistic quantum mechanics - the statement that $j_{\pm}^{0}$ is the charge density. As a result, it was further argued that $j_{\pm}^{0}$ disappears for neutral particles, for which the wave function had to be real. At the same time, no attention was paid to the fact that for free particles the requirement that the wave function be real means the rejection of their basic quantum properties. In particular, in the nonrelativistic limit, the KG equation turns into the Schrödinger equation, where the requirement that the wave function be real is simply unacceptable, since then, in particular, the usual time dependence and all wave properties of ensembles of particles disappear. For example, an ensemble of protons will interfere when passing through two slits, but the same ensemble of neutrons will not interfere, which is absurd. The absence of such a problem in SVM will also be discussed in the next section.

3.2. Wave equations with first-order time derivatives

In nonrelativistic quantum mechanics, the Schrödinger equation allows to calculate the wave function and its time derivative starting on the initial values of the wave function. In relativistic quantum mechanics, the wave equations (45) contain a second-order time derivative, and therefore at the initial moment it is necessary to specify initial values both wave function and its first order time derivative. This difference was considered one of the main weaknesses of the KG wave equation (45).

But later, representations of these wave equations were found in the form of two equations with first-order time derivatives (see [10]). Instead of the complex wave function and its time derivative, two functions φ and χ can be introduced:

$$\psi = \varphi + \chi, \quad i\hbar \frac{\partial \psi}{\partial t} = mc^2(\varphi - \chi). \tag{54}$$

Then the equation for ψ in (45) will turn into a system of two wave equations:

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \Delta(\varphi + \chi) + mc^2 \varphi, \tag{55}$$

$$-i\hbar \frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m} \Delta(\varphi + \chi) + mc^2 \chi. \tag{56}$$

This doubling of the number of equations has a fundamental reason and follows from the fact of doubling of the Hilbert space due to the presence of states of two energy signs describing particles and antiparticles. The nonrelativistic limit of these equations gives a separate Schrödinger equation for each sign of energy.

Therefore, the reason for the difference was that in nonrelativistic quantum mechanics, particles and antiparticles were described by the same wave equation for positive energies, whereas in TST, even in the nonrelativistic approximation, two Schrödinger equations are needed for states of two signs of energy.

Two equations (55)-(56) can be written as a single matrix equation with a combined wave function Ψ and Hamiltonian H:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi, \qquad \Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{57}$$

$$H = (\tau_3 + i\tau_2)\frac{\mathbf{p}^2}{2m} + \tau_3 mc^2, \tag{58}$$

where τ_i are isospin 2×2 matrices.

In this representation, the projection of the probability current density takes the form:

$$j^0 = j^0_+ + j^0_- = c(\varphi^*\varphi - \chi^*\chi) = c\Psi^*\tau_3\Psi. \tag{59}$$

Until now, this expression, due to the negative sign of the second term, was interpreted as an expression for the charge density and therefore for neutral particles it had to be equal to zero. In TST j^0 is a probability current density, and it is non-zero for all particles regardless of their charge, i.e. does not vanish for neutral particles either. From the probability current density, the interaction current density can also be formed by multiplying it the interaction constant, i.e. the value of charge, e. Then such an interaction current obviously vanishes for neutral particles, but not because of the disappearance of the probability current, which would be absurd, but because vanishing the interaction constant: $e = 0$.

3.3. Nonrelativistic limit of the relativistic wave equation

In wave equations (45), the nonrelativistic limit of positive energy states with asymptotic behavior is $\exp(-i\omega t)$ usually considered. Here we will consider this limit for states of both signs of energy $\psi = \psi_+ + \psi_-$ with asymptotics $\exp(\mp i\omega t)$.

In the nonrelativistic limit, the kinetic energy of the particle $\breve{E}_p = E_p - mc^2$ is small than the rest energy $\breve{E}_p \ll mc^2$ and therefore, in the time dependence of the wave function, a part depending on mc^2 oscillates rapidly. This high-frequency part can be separated from the slower part as:

$$\psi_+ = \breve{\psi}_+ \exp\left(-\frac{imc^2}{\hbar}t\right), \quad \psi_- = \breve{\psi}_- \exp\left(\frac{imc^2}{\hbar}t\right). \tag{60}$$

Here $\breve{\psi}_\pm$ are the slowly varying in time parts of the wave functions, for which:

$$\left|i\hbar\frac{\partial\breve{\psi}_\pm}{\partial t}\right| \approx \breve{E}_p\breve{\psi}_\pm \ll mc^2\breve{\psi}_\pm. \tag{61}$$

The first and second time derivatives $\psi_\pm$ with respect to time then take the form:

$$\frac{\partial\psi_\pm}{\partial t} = \left(\frac{\partial\breve{\psi}_\pm}{\partial t} \mp \frac{imc^2}{\hbar}\breve{\psi}_\pm\right)\exp\left(\mp\frac{imc^2}{\hbar}t\right) \approx \mp\frac{imc^2}{\hbar}\breve{\psi}_\pm \exp\left(\mp\frac{imc^2}{\hbar}t\right) \tag{62}$$

$$\begin{aligned}\frac{\partial^2\psi_\pm}{\partial t^2} &= \frac{\partial}{\partial t}\left[\left(\frac{\partial\breve{\psi}_\pm}{\partial t} \mp \frac{imc^2}{\hbar}\breve{\psi}_\pm\right)\exp\left(\mp\frac{imc^2}{\hbar}t\right)\right] \\ &\approx \left(\mp\frac{2imc^2}{\hbar}\frac{\partial\breve{\psi}_\pm}{\partial t} - \frac{m^2c^4}{\hbar^2}\breve{\psi}_\pm\right)\exp\left(\mp\frac{imc^2}{\hbar}t\right).\end{aligned} \tag{63}$$

Substituting them into the wave equation (45) gives two Schrödinger equations for two energy signs states:

$$i\hbar\frac{\partial\breve{\psi}_\pm}{\partial t} = \mp\frac{\hbar^2}{2m}\nabla^2\breve{\psi}_\pm \tag{64}$$

Let us consider the nonrelativistic limit for wave equations with the first order time derivatives only. For a free particle at rest there is no kinetic energy and the wave equations (55)-(56) and their solutions take the form:

$$i\hbar\frac{\partial\varphi}{\partial t} = mc^2\varphi, \quad \varphi(t) = a\cdot\exp\left(-\frac{i}{\hbar}mc^2t\right), \tag{65}$$

$$i\hbar\frac{\partial\chi}{\partial t} = -mc^2\chi, \quad \chi(t) = a\cdot\exp\left(\frac{i}{\hbar}mc^2t\right). \tag{66}$$

Therefore, in the nonrelativistic approximation for a moving particle we have:

$$\varphi = \breve{\varphi}\cdot\exp\left(-\frac{i}{\hbar}mc^2t\right), \quad \chi = \breve{\chi}\cdot\exp\left(\frac{i}{\hbar}mc^2t\right), \tag{67}$$

where $\breve{\varphi}$ and $\breve{\chi}$ are slowly evolving parts. From (55)-(56) then follow:

$$i\hbar\frac{\partial\breve{\varphi}}{\partial t} = -\frac{\hbar^2}{2m}\Delta\breve{\varphi} - \frac{\hbar^2}{2m}\Delta\breve{\chi}\cdot\exp\left(\frac{2i}{\hbar}mc^2t\right), \tag{68}$$

$$-i\hbar\frac{\partial\breve{\chi}}{\partial t} = -\frac{\hbar^2}{2m}\Delta\breve{\chi} - \frac{\hbar^2}{2m}\Delta\breve{\varphi}\cdot\exp\left(-\frac{2i}{\hbar}mc^2t\right). \tag{69}$$

In these equations, in the nonrelativistic region, it is necessary to average over the high-frequency contributions in the time interval $\Delta t_{усред}$, which is quite small compared to the periods of the processes under study, but quite large compared to the Compton period: $\hbar / \breve{E}_p \gg \Delta t_{усред} \gg \hbar / mc^2$. Then the rapidly oscillating exponentials in (68)-(69) make a practically vanishing contribution and these equations turn into two Schrödinger equations that describe two kind of particles with opposite signs on energy:

$$i\hbar \frac{\partial \breve{\varphi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \breve{\varphi}, \quad i\hbar \frac{\partial \breve{\chi}}{\partial t} = \frac{\hbar^2}{2m} \Delta \breve{\chi}. \tag{70}$$

4. Time-symmetric relativistic quantum mechanics of spin ½ particle

4.1. Justification for the transition from KG to Dirac equation

In relativistic quantum mechanics, the transition from KG to Dirac equation was justified by the need to solve two problems:

a) KG equation is second order with respect to the time derivative and it is necessary to set the initial values of both the wave function and its time derivative;

b) the bilinear expression (51) is not positive definite and therefore cannot be directly interpreted as a probability density.

The first problem was solved at early periods - the KG equation was written in the form of two equations with first-order derivatives (see [10] and section 3.2) and this system of equations for a scalar particle is similar to the Dirac equation. The doubling of the equations is associated with the doubling of the Hilbert state in the relativistic theory due to the presence of negative energy particles describing antiparticles.

In TST, the second problem also disappears, since the bilinear expression (51) describes not the probability density, but the probability current density along the time axis and the sign of this current indicates the opposite evolving of particles of different energy signs.

As a result, the only difference between the KG and Dirac equations is that the latter is a system of equations for *spinor* wave functions for particles of spin ½, which transforms into the Pauli equation in the nonrelativistic limit.

Thus, the difference between the KG and Dirac equations is reduced only to the difference in the spins of the particles whose state they describe. Each of these equations describes particles of two energy signs, both have correct non-relativistic limits and they have no problems with probabilistic interpretation of wave functions.

Further in this part of the article some corrections that TST introduces to the theory of spin ½ particles will be considered.

4.2. Dirac equations for positive energy particles and their solutions

In the theory of spin ½ particles, the difference between TST and the previous standard formulation, as in the case of a scalar particle, is the emphasis on the differences between the probability density and the probability flux density under the Lorentz and inversion transformations. This allows one to correctly interpret the negative sign of the probability current for negative energies. Therefore, here standard relations for positive energy will be presented, which will be used in the formulation of a consistent theory of negative energy particles.

In the inertial frame K_+, the basis of which consists of positive energy particles, the wave functions ψ_+^r and $\bar{\psi}_+^r \equiv \psi_+^{r+} \gamma^0$ ($r = 1, 2$) for positive energy particles of spin ½ are bispinors and satisfy the Dirac equations:

$$i\gamma^\mu \partial_\mu \psi_+^r - \kappa \psi_+^r = 0, \tag{71}$$

$$i\partial_\mu \bar{\psi}^r_+ \gamma^\mu + \kappa \bar{\psi}^r_+ = 0. \tag{72}$$

At turning from K to another frame K', the coordinates, wave functions and Dirac matrices are transformed as:

$$x^\mu{}' = a^\mu_\nu x^\nu, \tag{73}$$

$$\psi'(x') = S(a)\psi(x), \tag{74}$$

$$S^{-1}(a)\gamma^\mu S(a) = a^\mu_\nu \gamma^\nu, \tag{75}$$

where the matrix S is equal to

$$S(a) = \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu} I^{\mu\nu}_n\right). \tag{76}$$

Here ω is the "rotation angle" around the axis with direction n, $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$ and $I^{\mu\nu}_n$ is the rotation matrix. In the general case, and in particular at Lorentz transformations, the relation $S^{-1} = \gamma^0 S^+ \gamma^0$ holds, and the matrix S is unitary $S^{-1} = S^+$ only under transformations of spatial coordinates only, including their inversion.

The simplest solutions to the equations (71)-(72) are plane waves:

$$\psi^r_+ = u^r(p)\exp(-ip_\mu x^\mu / \hbar). \tag{77}$$

In this case, bispinors $u^r(p_0, \mathbf{p})$ are easiest to find in the rest frame K'_+ comoving the particle. In this frame $u^r(mc, 0)$, and the equations (71)-(72) take a simple form:

$$i\gamma^0\partial_0\psi^r_+ = \kappa\psi^r_+, \quad i\partial_0\bar{\psi}^r_+\gamma^0 = -\kappa\bar{\psi}^r_+, \tag{78}$$

and their solutions are equal to:

$$\psi^r_+ = u^r(0)\exp(-imc^2 t/\hbar), \tag{79}$$

$$u^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{80}$$

Further, the Lorentz transformations on this wave functions, by using (73)-(76), give their general form (77) in the frame K_+ where this particle is moving. As a result of these transformations, bispinors $u^r(p)$ take the form:

$$u^1(p) = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} 1 \\ 0 \\ p_z/\tilde{p} \\ p_+/\tilde{p} \end{pmatrix}, \quad u^2(p) = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} 0 \\ 1 \\ p_-/\tilde{p} \\ -p_z/\tilde{p} \end{pmatrix}, \tag{81}$$

where $\tilde{p} = mc + E_p/c$, $p_\pm = p_1 \pm ip_2$. They satisfy the relations:

$$\bar{u}^r(p)u^r(p) = \delta^{rr'}, \tag{82}$$

$$\bar{u}^r(p)\gamma^0 u^r(p) = u^{r+}(p)u^r(p) = \delta^{rr'} E_p / mc^2. \tag{83}$$

which gives normalization conditions for the wave functions (77):

$$\bar{\psi}^r_+ \psi^{r'}_+ = \delta^{rr'}, \tag{84}$$

$$\bar{\psi}^r_+ \gamma^0 \psi^{r'}_+ = \psi^{r+}_+ \psi^{r'}_+ = \delta^{rr'} E_p / mc^2. \tag{85}$$

To express the conserved 4-vector of probability current j^μ in terms of ψ and $\bar{\psi}$ the equation (71) should be multiplied on the left by $c\bar{\psi}$ and (72) on the right by $c\psi$, then the results should be added:

$$c\bar{\psi}\gamma^\mu \partial_\mu \psi + c(\partial_\mu \bar{\psi})\gamma^\mu \psi = \partial_\mu (c\bar{\psi}\gamma^\mu \psi) = \partial_\mu j^\mu = 0, \tag{86}$$

$$j^\mu = c\bar{\psi}\gamma^\mu \psi. \tag{87}$$

It follows from (85) that the 4-component of the probability flux for positive energy particles is also positive defined: $j^0_+ = c\psi^+_+ \psi_+ = cu^+ u \geq 0$.

4.3. Dirac equations for negative energy particles and their solutions

In the frame K_- the basis consists of negative energy particles, and the coordinates in this frame are related to the coordinates K_+ by 4-inversion operation TP:

$$x^\mu_- = a^\mu_\nu x^\nu = -x^\mu, \quad a^\mu_\nu = -\delta^\mu_\nu. \tag{88}$$

The wave functions for negative energy particles and spin ½ are bispinors, satisfying the same Dirac equations as in (71)-(72) ($r = 3,4$):

$$i\gamma^\mu \partial_{\mu-} \psi^r_- - \kappa \psi^r_- = 0, \tag{89}$$

$$i\,\partial_{\mu-} \bar{\psi}^r_- \gamma^\mu + \kappa \bar{\psi}^r_- = 0. \tag{90}$$

Since at 4-inversion $\partial_\mu \to \partial_{\mu-} = -\partial_\mu$, the spinor transformation matrix TP must anticommute with γ^μ, we have for this matrix:

$$TP = i\gamma^1\gamma^2\gamma^3\gamma^0 = \gamma^5 \tag{91}$$

$$\psi'(x') = PT\psi(x) = \gamma^5 \psi(x), \tag{92}$$

$$(PT)^{-1}\gamma^\mu PT = \gamma^5 \gamma^\mu \gamma^5 = -\gamma^\mu. \tag{93}$$

From (91), taking into account $P = \gamma^0$, it follows that the time axis inversion matrix T is:

$$T = i\gamma^1\gamma^2\gamma^3. \tag{94}$$

In the previous standard formulation, the time axis inversion operation was divided into two stages. This was charge conjugation $\psi_c = i\gamma^2 \psi^*$, consisting of complex conjugation and multiplication by $C = i\gamma^2$, and the Wigner time reversal with matrix $T_W = i\gamma^1\gamma^3$ and also with complex conjugation $\psi(t') = i\gamma^1\gamma^3 \psi^*(t)$. As a result, the complex conjugation, performed twice, was eliminated and the time axis inversion was obtained, as in (94):

$$T = iCT_W. \tag{95}$$

When the time axis is inverted, instead of complex conjugation, the initial and final states are rearranged, since the previous evolution occurs in the opposite direction to the new axis.

At turning from K_- to another frame, K'_- the coordinates, wave functions and Dirac matrices are transformed in the same way as in (73)-(76). Here the 4-inversion is added only when transforming wave functions from K_- to K_+ and vice versa.

The simplest solutions to the equations (89)-(90) are plane waves:

$$\psi_-^r = u_-^r(p_{(-)})\exp(-ip_{\mu-}x_-^\mu / \hbar). \tag{96}$$

Bispinors $u_-^r(p_0, \mathbf{p})$ are first found in the rest frame K'_-, which comoves the particle and also moves in the opposite direction in time compared to K_+. In this frame $u_-^r(mc, 0)$, the wave functions have the same form as in (79)-(80), but with the replacement $t \to t_-$.

Then, having carried out Lorentz transformations according to (73)-(76) over the wave function in the rest frame of the particle K'_-, we will find its form in the inertial frame K_- where this particle is moving. As a result, the bispinors $u^r(p)$ take the form (96), where $u_-^r(p_{(-)})$ have the form (81), but with the replacement of momentum $p \to p_{(-)}$. The same form of wave functions in two frames K_+ and K_- expresses the symmetry between particles of two energy signs.

Further, to describe the interactions of particles of both signs of energy, it is necessary to consider them in the same frame and it is natural to choose K_+. Therefore, the next step is to convert the wave functions ψ_- from the coordinates K_- to the coordinates K_+. To do this, we perform a 4-inversion according to (88), (91)-(93), also changing the sign of 4-momentum $p_{(-)}^\mu \to -p^\mu$. In this case, two parts of the wave function with different signs of energy are written for the same world point and therefore the product $-ip_{\mu-}x_-^\mu$ turns to $ip_\mu x^\mu$. As a result, we get:

$$\gamma^5\psi_-(x_-) = \psi_-(x_+) = v^r(-p)\exp(ip_\mu x^\mu / \hbar). \tag{97}$$

For clarity, we present the transformation of the spinor part of the wave function:

$$\gamma^5 u_-^1(p_{(-)}) = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -p_z / \tilde{p} \\ -p_+ / \tilde{p} \end{pmatrix} = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} -p_z / \tilde{p} \\ -p_+ / \tilde{p} \\ 1 \\ 0 \end{pmatrix} = v^1(-p), \tag{98}$$

$$\gamma^5 u_-^2(p_{(-)}) = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -p_- / \tilde{p} \\ p_z / \tilde{p} \end{pmatrix} = \sqrt{\frac{\tilde{p}}{2mc}} \begin{pmatrix} -p_- / \tilde{p} \\ p_z / \tilde{p} \\ 0 \\ 1 \end{pmatrix} = v^2(-p). \tag{99}$$

At 4-inversion, the scalar $\bar{\psi}_-^r(x_-)\psi_-^{r'}(x_-)$ must not change, whereas the 4-vector $\bar{\psi}_-^r(x_-)\gamma^\mu\psi_-^{r'}(x_-)$ must change sign. However, the Lorentz transformations of bispinors are not unitary and, as the result, these bilinear forms include not the Hermitian conjugate, but the

Dirac conjugate bispinor. At 4-inversion, this leads to an additional change in sign and the result is expressions with opposite signs compared to the expected ones:

$$u_-^{r+}(p_{(-)})\gamma^5 \cdot \gamma^0 \cdot \gamma^5 u_-^{r'}(p_{(-)}) = -\bar{v}^{r'}(-p)v^r(-p), \tag{100}$$

$$u_-^{r+}(p_{(-)})\gamma^5 \cdot \gamma^5 u_-^{r'}(p_{(-)}) = v^{r'+}(-p)v^r(-p). \tag{101}$$

Here the sign of a scalar changes, while the sign of a vector component does not change.

To correct this shortcoming and restore the normal properties of scalars and vectors at 4-inversion, we must return to the general definition of the average of the operator A, which does not depend on a specific normalization agreement:

$$\langle A \rangle = \frac{\bar{\psi} A \psi}{\bar{\psi}\psi}. \tag{102}$$

With this more rigorous definition of means, expressions for bilinear forms of bispinors take the form required by their properties under 4-inversion:

$$\frac{\bar{\psi}_-^r \psi_-^{r'}}{\bar{\psi}_-^r \psi_-^{r'}} \to \frac{\bar{v}^r \gamma^0 v^{r'}}{\bar{v}^r v^{r'}} = \delta^{rr'}. \tag{103}$$

$$\frac{\bar{\psi}_-^r \gamma^0 \psi_-^{r'}}{\bar{\psi}_-^r \psi_-^{r'}} \to \frac{\bar{v}^r \gamma^0 v^{r'}}{\bar{v}^r v^{r'}} = -\frac{E}{mc^2}\delta^{rr'}. \tag{104}$$

Thus, the K_+ 4-component of the probability flux density for negative energy particles, similar to the case of bosons, is negative definite $j_-^0 \le 0$.

4.4. Wave equations for spinors and the nonrelativistic limit

The wave function can be written as a bispinor, composed of two spinors φ and χ:

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{105}$$

after which Eq. (45) transforms into a system of two spinor equations:

$$i\hbar \frac{\partial \varphi}{\partial t} = -i\hbar \boldsymbol{\gamma} \cdot \nabla \chi + mc^2 \varphi, \tag{106}$$

$$-i\hbar \frac{\partial \chi}{\partial t} = -i\hbar \boldsymbol{\gamma} \cdot \nabla \varphi + mc^2 \chi. \tag{107}$$

In this representation, the probability flux density takes the form:

$$j^0 = c\tilde{\psi}\gamma^0\psi = c(\varphi^+\varphi - \chi^+\chi). \tag{108}$$

Here the negative sign of the second term expresses the opposite direction of the probability current along the time axis for the contribution of negative energy states that arises at the Lorentz transformations.

In wave equations (71) the nonrelativistic limit of positive energy states is usually considered. Here we will consider this limit for the states of both signs of energy.

In the nonrelativistic limit, the kinetic energy of the particle $\tilde{E}_p = E_p - mc^2$ is small compared to the rest energy $\tilde{E}_p \ll mc^2$. Therefore, in the time dependence of wave function, $\psi(x)$ the part depending on mc^2 oscillates quickly and this part can be separated from the slowly varying part. To do this, we separate in $\psi(x)$ two components $\psi_\pm$ corresponding to states of different signs of energy:

$$\psi_+ = \tilde{\psi}_+ \exp\left(-\frac{imc^2}{\hbar}t\right), \quad \psi_- = \tilde{\psi}_- \exp\left(\frac{imc^2}{\hbar}t\right). \tag{109}$$

Here $\tilde{\psi}_\pm$ are slowly varying parts of the wave function, for which:

$$\left| i\hbar \frac{\partial \tilde{\psi}_\pm}{\partial t} \right| \approx \tilde{E}_p \tilde{\psi}_\pm \ll mc^2 \tilde{\psi}_\pm \tag{110}$$

The time derivatives $\psi_\pm$ are then equal to:

$$\frac{\partial \psi_\pm}{\partial t} = \left(\frac{\partial \tilde{\psi}_\pm}{\partial t} \mp \frac{imc^2}{\hbar} \tilde{\psi}_\pm \right) \exp\left(\mp \frac{imc^2}{\hbar} t \right) \approx \mp \frac{imc^2}{\hbar} \tilde{\psi}_\pm \exp\left(\mp \frac{imc^2}{\hbar} t \right) \tag{111}$$

Substituting them into the wave equations (71) gives two Schrödinger equations for states of two energy signs:

$$\pm i\hbar \frac{\partial \tilde{\psi}_\pm}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \tilde{\psi}_\pm \tag{112}$$

For a free particle at rest there is no kinetic energy and the wave equations (106)-(107) and their solutions are extremely simplified:

$$i\hbar \frac{\partial \varphi}{\partial t} = mc^2 \varphi, \quad \varphi(t) = a \cdot \exp\left(-\frac{i}{\hbar} mc^2 t \right), \tag{113}$$

$$i\hbar \frac{\partial \chi}{\partial t} = -mc^2 \chi, \quad \chi(t) = a \cdot \exp\left(\frac{i}{\hbar} mc^2 t \right). \tag{114}$$

Therefore, in the nonrelativistic approximation for a moving particle we have:

$$\varphi = \tilde{\varphi} \cdot \exp\left(-\frac{i}{\hbar} mc^2 t \right), \quad \chi = \tilde{\chi} \cdot \exp\left(\frac{i}{\hbar} mc^2 t \right), \tag{115}$$

where $\tilde{\varphi}$ and $\tilde{\chi}$ are the slowly varying parts of the wave function. Eqs. (106)-(107) then give:

$$i\hbar \frac{\partial \tilde{\varphi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \tilde{\varphi} - \frac{\hbar^2}{2m} \Delta \tilde{\chi} \cdot \exp\left(\frac{2i}{\hbar} mc^2 t \right), \tag{116}$$

$$-i\hbar \frac{\partial \tilde{\chi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \tilde{\chi} - \frac{\hbar^2}{2m} \Delta \tilde{\varphi} \cdot \exp\left(-\frac{2i}{\hbar} mc^2 t \right). \tag{117}$$

In these equations, in the nonrelativistic region, time averaging is performed in the interval $\Delta t_{усред}$, which is quite small compared to the periods of the considered processes, but quite large compared to the Compton period for the particle: $\hbar / \tilde{E}_p \gg \Delta t_{усред} \gg \hbar / mc^2$. Then the oscillating exponentials in (68)-(69) make a practically vanishing contribution and the equations turn into two Schrödinger equations that describe particles of two energy signs:

$$i\hbar \frac{\partial \tilde{\varphi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \tilde{\varphi}, \quad -i\hbar \frac{\partial \tilde{\chi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \tilde{\chi}. \tag{118}$$

4.5. Causal propagators of particles

The causal propagator $G_c(x'-x)$ is the probability amplitude of the transition of a scalar particle from the initial event x to the final event x' and satisfies KG equation with a point source:

$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)G_c(x'-x) = -\delta^4(x'-x). \tag{119}$$

Solving this equation in momentum representation gives an expression for the momentum representation of the propagator, satisfying the boundary conditions of ZSF interpretation:

$$G_c(x'-x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x'-x)}}{p^2 - m^2c^2 + i\varepsilon}. \tag{120}$$

This propagator gives non-zero transition probabilities for positive energy particles at $t' > t$, and for negative energy particles at $t' < t$.

The causal propagator $S_c(x'-x)$ for a spin ½ particle satisfies the Dirac equation with a point source:

$$(i\hbar\gamma^\mu\partial_\mu - mc)S_c(x'-x) = \delta^4(x'-x). \tag{121}$$

The solution of this equation with the boundary conditions of ZSF in the momentum representation has the form:

$$S_c(x'-x) = \int \frac{d^4p}{(2\pi)^4}\frac{\gamma^\mu p_\mu + mc}{p^2 - m^2c^2 + i\varepsilon}e^{-ip(x'-x)}. \tag{122}$$

This propagator also gives non-zero transition probabilities for positive energy particles at $t' > t$, and for negative energy particles at $t' < t$.

In quantum field theory with only physical states of positive energy, the use of these expressions for propagators containing poles at negative p_0 was considered as a purely formal tool. In TST, the presence of such a pole is natural and does not cause any contradictions.

In the TST, the inclusion of interactions, in particular, the electromagnetic field, as well as the formulation of the diagram technique, is ultimately carried out in the same way as in standard courses of relativistic quantum mechanics [4,5].

5. Conclusion

TST based on ZSF interpretation includes particles of both signs of energy, moving in mutually opposite directions of the time axis of the ordinary inertial frame K_+, where the negative energy particles are a covariant way of describing phenomena with antiparticles.

In TST not only the negative energy particles move backward in the ordinary time of K_+, but their rest frames K_- also move backward in this time. This leads to a corresponding extension of the Poincaré group. The connection between two classes of transformations for two directions of time evolution is carried out by 4-inversion PT.

At the same time, due to the absence of antiparticles in TST, the time axis inversion operation T is a combination of the previous operations of the Wigner time reversal T_W and charge conjugation C.

The time integration in the action function also occurs in both directions of time. Writing all time integrals through forward time integrals makes the formulas more compact.

In TST there are no problems with the probabilistic interpretation of the wave functions of both scalar particles and fermions, the theories of which are introduced necessary corrections making them physically consistent. TST also leads naturally to the standard causal propagators of particles.

Since previous physical theories became the basis of the physical picture of the world and were widely used for practical purposes long before the discovery of antiparticles, the appearance and successes of ZSF interpretation of the did not affect either the presentation of their formalism in standard textbooks or their use in the scientific literature. The formulation of TST will lead to the development of a new generation of scientific and educational literature containing presentations of a more consecutive and consistent, from a physical point of view, description of phenomena in systems with antiparticles.

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