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Metrics with irreducible mass leading to a correct parameter dependence of gravitational effects around charged and rotating bodies

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Abstract

The standard metrics outside of charged and rotating sources (Kerr-Newman in general, Reissner-Nordström and Kerr in particular) contain the total mass at infinity, which includes both the mass of a neutral non-rotating body and the mass equivalents of rotational and electric field energies. Therefore, the total mass is related with other parameters of the metric - angular momentum and charge. However, at studying the dependence of gravitational effects on parameters, this relationship was ignored, assuming total mass to be constant at varying angular momentum and charge. This error led to physically absurd predictions about the weakening of gravity and its effects with increasing the rotational and electric field energies. To eliminate such errors, the total mass must be expressed in terms of independent parameters - the mass of the neutral non-rotating matter of the source, charge and angular momentum. Recently this has been done using as an independent parameter the mass determined from the gravitational radius at the pole when the charge is only on the surface (Zakir, 2022). In the present paper it is used "irreducible mass", earlier defined heuristically as the remainder of total mass after the removal of angular momentum and charge. Earlier, the mass formulas expressing total mass in terms of irreducible mass were proposed by Florides (1960) (improved by the author (2022)) for charged bodies and then by Christodolou (1970) for rotating and Christodolou-Ruffini (1971) for charged rotating sources. In the paper, the standard metrics are transformed to metrics with independent parameters by substituting the expression for the total mass according to these mass formulas. It is shown that the metrics in this form lead to a physically correct dependence of the effects of gravity on the parameters, in particular, the growth of the rotation and electric field energies strengthens gravity and its effects, such as time dilation and redshifts, increases radii of orbits and the area of shadow.

Keywords: total mass, charge, angular momentum, Reissner-Nordström metric, Kerr metric, Kerr-Newman metric, mass formula, redshifts, photon orbit, orbits of particles, shadow

Content

1. Introduction

Standard metrics outside sources of gravity, Reissner-Nordström (RN) for a charged, Kerr for a neutral rotating and Kerr-Newman (KN) for a charged rotating one, being exact solutions

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of the Einstein equations, contain as the main parameter the total weight the total mass of the source M , determined from the coincidence with the Schwarzschild metric at $r \to \infty$ [1-5].

But M , the mass equivalent of the total energy in a sphere of infinite radius, contains as the mass of a neutral non-rotating source M_0 , and the mass equivalents of the rotational and the electric field energies. Therefore, M depends on other parameters of the metrics – the rotation parameter a (angular momentum per unit mass) and the charge Q of the source.

However, at studying the dependence of gravitational effects on parameters, this fact was ignored, assuming $M = const$ at varying a and Q . The result of this mistake was physically absurd predictions about the weakening of gravity and its effects with the addition of the rotational and electric field energies [2-9].

In fact, M is an integration constant with respect to coordinates only, but is a function of independent parameters M_0 , a and Q . Thus, the standard metrics were solutions up to some function $M(M_0, a, Q)$. Therefore, in order to completely solve the problem, excluding the above errors, it was necessary to find an explicit form of this function, after which the substitution of the expression M through M_0 , a and Q into the standard metrics transforms them into *the metrics with independent parameters*.

The problem of finding metrics with independent parameters was firstly formulated and solved in the simplest cases in the previous papers [10,11] (see also [12]). In the KN metric, the mass M_0 of a source with a charge on the surface, in particular, on the surface of infinite redshift, was identified with the effective mass M_+ defining from the gravitational radius at the pole $r_+ = 2M_+ = M + \sqrt{M^2 - a^2 - Q^2}$. By expressing M through M_+ , a and Q , it was found the simple mass formula:

$$M = M_+ + \frac{a^2 + Q^2}{4M_+}. \tag{1}$$

The choice $M_0 \simeq M_+$ and substitution of (1) into the KN metric led to the physically correct predictions about strengthening of gravity and its effects with growth a and Q .

However, independence M_+ from Q was clear for charged non-rotating sources only, while independence from a was only assumed and was not proven. In this regard, it is of interest to study other possibilities for determining M_0 .

In the present paper it will be studied the consequences of the identification $M_0 \simeq M_{ir}$, where M_{ir} is "irreducible mass", usually defined heuristically (using specific processes) as the remainder of the total mass after removal of angular momentum and charge. Mass formulas expressing M through M_{ir} were proposed by Florides (1960) [13] (improved by the author (2022) [10]) for charged bodies, Christodolou (1970) for rotating [14] and Christodolou-Ruffini (1971) for rotating charged sources [15]. Below it will be used the Christodolou-Ruffini mass formula [12]:

$$M^2 = \left(M_{ir} + \frac{Q^2}{4M_{ir}} \right)^2 + \frac{J^2}{4M_{ir}^2}. \tag{2}$$

Substituting the expression for the source's angular momentum $J = Ma$ allows us to write the mass formula (2) in the form $M = f(M_{ir}, a, Q)$, more convenient for applications:

$$M = M_{ir}\frac{1+Q^2/4M_{ir}^2}{\sqrt{1-a^2/4M_{ir}^2}}. \tag{3}$$

At $a=0$, two formulas (1) and (3) coincide exactly, and in this case $M_{ir} = M_+^{(Q)}$, where $2M_+^{(Q)} = M + \sqrt{M^2 - Q^2}$. At $a^2 > 0$, one has $M_{ir}^2 = M_+^2 + a^2/4 > M_+^2$, and at $a \to 2M_{ir}$ the dependence M on a in (3) is singular, while in (1) it is regular.

Substitution of Eq. (3) into the standard KN metric transforms it into a metric with independent parameters M_{ir}, a and Q. At studying the dependence on a and Q of the properties of compact sources and the effects of gravity around them, this form of the KN metric is more correct, both from mathematical and physical points of view.

In the paper the theoretical and observational consequences of this new form of the KN metric are considered. In Sections 2 and 3, the consequences of new forms of particular cases, the RN and Kerr metrics, are presented, and in Section 4, the full KN metric is discussed.

2. Metric with irreducible mass for charged bodies and its consequences

2.1. The Reissner-Nordström metric in terms of irreducible mass

The space-time interval outside a spherically symmetric body with a charge Q in the rest frame of its center is given by the RN metric:

$$ds^2 = g_{00}(r)c^2dt^2 + g_{11}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

$$g_{00} = -g_{11}^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \equiv f(r). \tag{5}$$

Here the total mass M coincides with the Schwarzschild mass in the asymptotics $r \to \infty$ and therefore includes not only the mass of a neutral body M_0 (which in this paper we will identify with irreducible mass $M_0 = M_{ir}$), but also the mass equivalent of the electric field energy (proportional to Q^2) in a sphere of infinite radius. Therefore, M depends on Q and we find this dependence from the mass formula (3) for $J = a = 0$:

$$M = M_{ir} + \frac{Q^2}{4M_{ir}}. \tag{6}$$

Substitution (6) into (5) gives for the time and radial components of the metric:

$$g_{00} = -g_{11}^{-1} = 1 - \frac{2M_{ir} + Q^2/2M_{ir}}{r} + \frac{Q^2}{r^2} = 1 - \frac{2M_{ir}}{r} - \frac{Q^2}{2M_{ir}r}\left(1 - \frac{2M_{ir}}{r}\right), \tag{7}$$

or in a more compact form:

$$g_{00} = -g_{11}^{-1} = \left(1 - \frac{2M_{ir}}{r}\right)\left(1 - \frac{Q^2}{2M_{ir}r}\right). \tag{8}$$

Thus, the RN metric in terms of irreducible mass (7)-(8) transforms to the metric with independent parameters M_{ir} and Q. In the region $r > 2M_{ir} > Q$, this metric has no singularities. In the standard RN metric (5), the sign of the electric term is positive $+Q^2/r^2$ and opposite to the sign of the Schwarzschild term $-2M/r$, and therefore the growth of Q under the condition $M = const$ increases g_{00}, which means the weakening of gravity and its effects [6,7].

On the contrary, in the RN metric with independent parameters (7), containing the independent parameter M_{ir} instead of the dependent one M, the bracket in (7) is positive outside the source $r > 2M_{ir} > Q$ and, therefore, the contribution to the metric of the electric field energy, given by the term with Q^2, has the same sign as the contribution of the neutral matter energy given by $2M_{ir}/r$. As a result, growth of Q strengthens gravity and its observable effects, and these physically correct dependences on Q are given in the Section 2.2.

For a neutral test particle moving on the equatorial plane and having the mass μ, the energy E and the angular momentum L, the equation of motion has the form:

$$\dot{r}^2 = \frac{E^2}{\mu^2} - V^2, \tag{9}$$

where the effective potential V, taking into account (8), has the form ($k = 0, \pm 1$):

$$V^2 = \left(k + \frac{L^2}{\mu^2 r^2}\right)\left(1 - \frac{2M_{ir}}{r}\right)\left(1 - \frac{Q^2}{2M_{ir} r}\right). \tag{10}$$

Notice that the growth of Q lowers the effective potential, increasing the gravitational attraction to the source, and at the same time, the contribution to V of the electric field energy has the same form as the contribution of the rest energy of a neutral body.

2.2. Correct dependence of gravitational effects on source charge

The proper time interval between two events $\Delta\tau = g_{00}^{1/2}(r)\Delta t$ for a test particle that is at rest near the source at the point $(r, \theta, \varphi) = const$ is less than the interval Δt between the same events according to distant clocks. This is the gravitational slowdown of proper times which for two forms of the RN metric (5) and (8) is given by:

$$\Delta\tau^{(a)} = \Delta t\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}, \tag{11}$$

$$\Delta\tau^{(b)} \simeq \Delta t\left(1 - \frac{2M_{ir}}{r}\right)^{1/2}\left(1 - \frac{Q^2}{2M_{ir} r}\right)^{1/2}. \tag{12}$$

As shows Fig. 1, the growth of Q in (11) at the condition $M = const$ leads to an increase in $g_{00}^{1/2}(r)$, which means the *weakening* of gravity and the time dilation effect. The opposite situation takes place in (12), where the growth of Q decreases $g_{00}^{1/2}(r)$, which means the *strengthening* of gravity and the time dilation effect. In this figure and below M, Q and a are given in units M_{ir}.

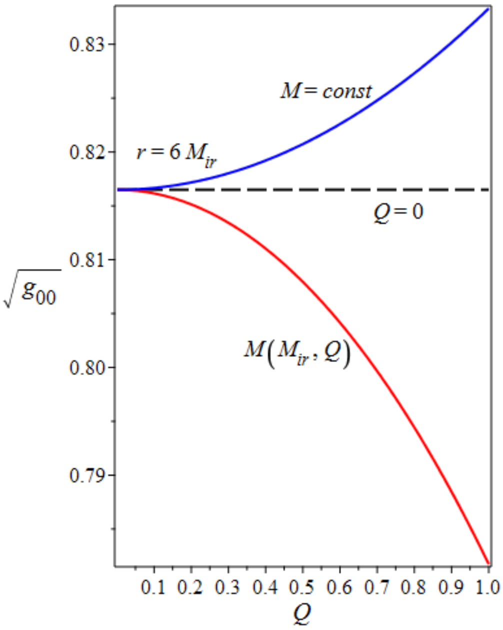

Fig. 1. Dependence on Q of time dilation, $r = 6M_{ir}$.

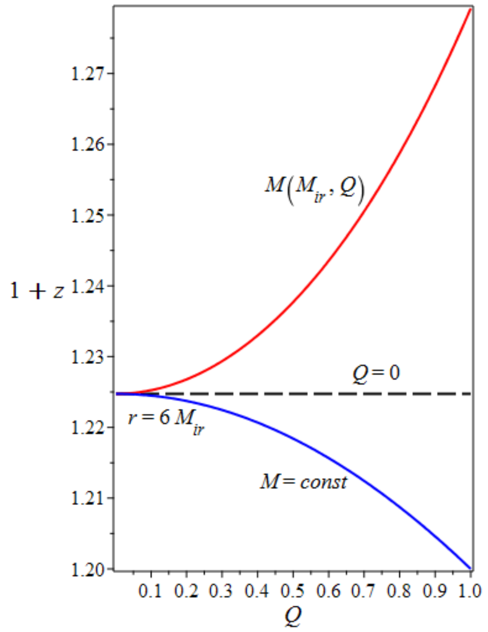

Fig. 2. Dependence on Q of redshift, $r = 6M_{ir}$.

The gravitational redshift follows from the gravitational slowdown of proper time. Frequency ω (in terms of t) of photons from a source resting at a point (r,θ,φ) near the collapsar, when received far away, there will be less than frequency ω_0 of local photons emitted near the observer: $\omega < \omega_0$. The redshift factor $1+z = \omega_0 / \omega = g_{00}^{-1/2}$ in the two cases under consideration is given by the expressions:

$$1+z^{(a)} = \frac{\omega_0}{\omega} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2}, \tag{13}$$

$$1+z^{(b)} = \frac{\omega_0}{\omega} = \left(1 - \frac{2M_{ir}}{r}\right)^{-1/2}\left(1 - \frac{Q^2}{2M_{ir}r}\right)^{-1/2}. \tag{14}$$

As shows Fig. 2, with growth in Q the gravitational redshift decreases in the case (13) and increases in the case (14).

Equations of motion in the RN metric (5) and their solutions for geodesic trajectories, especially for several distinguished orbits and a shadow, are well known (see [1–7]). Corresponding formulas for the new form of this metric with independent parameters (8) follow from them when substituting instead of M its expression through M_{ir} and Q in (6). For this reason, it is sufficient to present only the results for the former and new dependences on Q.

The formulas for the radius of the photon orbit, $r_{ph}^{(a)}$ in terms of M and then $r_{ph}^{(b)}$ in terms of M_{ir}, are:

$$r_{ph}^{(a)} = \frac{3}{2}M\left(1 + \sqrt{1 - \frac{8Q^2}{9M^2}}\right), \tag{15}$$

$$r_{ph}^{(b)}=\frac{3}{2}\left(M_{ir}+\frac{Q^2}{4M_{ir}}\right)\left(1+\sqrt{1-\frac{8Q^2}{9(M_{ir}+Q^2/4M_{ir})^2}}\right). \tag{16}$$

For the radius of the innermost stable circular orbit of a massive particle r_{ISCO} we have:

$$r_{ISCO}^{(a)}=M\left[2+\xi+\frac{1}{\xi}\left(4-\frac{3Q^2}{M^2}\right)\right], \tag{17}$$

$$r_{ISCO}^{(b)}=\left(M_{ir}+\frac{Q^2}{4M_{ir}}\right)\left[2+\xi+\frac{1}{\xi}\left(4-\frac{3Q^2}{(M_{ir}+Q^2/4M_{ir})^2}\right)\right], \tag{18}$$

$$\xi=\left[8+\frac{2Q^4}{M^4}-\frac{Q^2}{M^2}\left(9-\sqrt{\left(1-\frac{Q}{M}\right)\left(5-\frac{4Q}{M}\right)}\right)\right]^{1/3}. \tag{19}$$

In this case, in the formula (18) for ξ from , (19) its expression is substituted (6) for M .

For the radius of the shadow of a charged collapsar $\lambda_c=r_{ph}g_{00}^{-1/2}(r_{ph})$, by substituting g_{00} from (5) and (8) with r_{ph} from (15)-(16), we obtain the formulas:

$$\lambda_c^{(a)}=\frac{3}{2}M\cdot\frac{1+\sqrt{1-8Q^2/9M^2}}{\sqrt{1-2M/r_{ph}+Q^2/r_{ph}^2}}. \tag{20}$$

$$\lambda_c^{(b)}=\frac{3}{2}\left(M_{ir}+\frac{Q^2}{4M_{ir}}\right)\cdot\frac{1+\sqrt{1-8Q^2/9(M_{ir}+Q^2/4M_{ir})^2}}{\sqrt{(1-2M_{ir}/r_{ph})(1-Q^2/2M_{ir}r_{ph})}}. \tag{21}$$

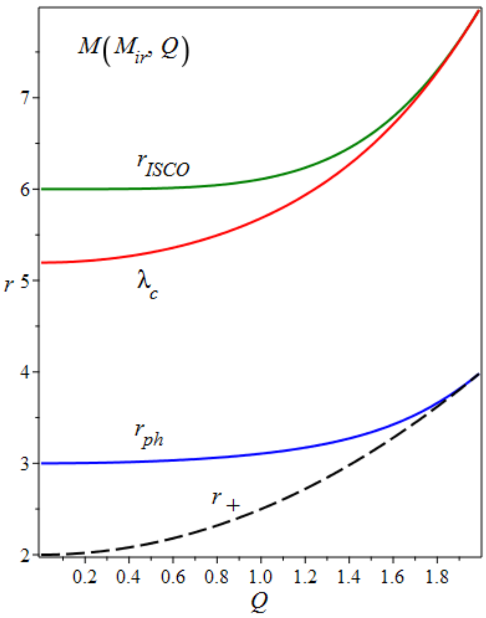

Fig. 3. Charge dependence of radii of orbits and shadow in the RN metric with irreducible mass.

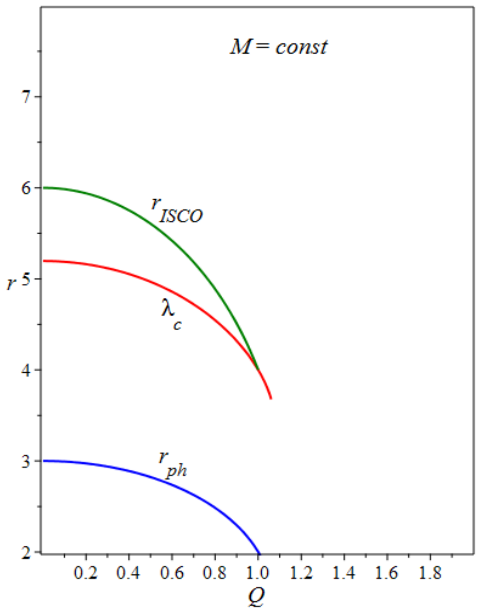

Fig. 4. Charge dependence of radii of orbits and shadow in the standard RN metric and $M = const$.

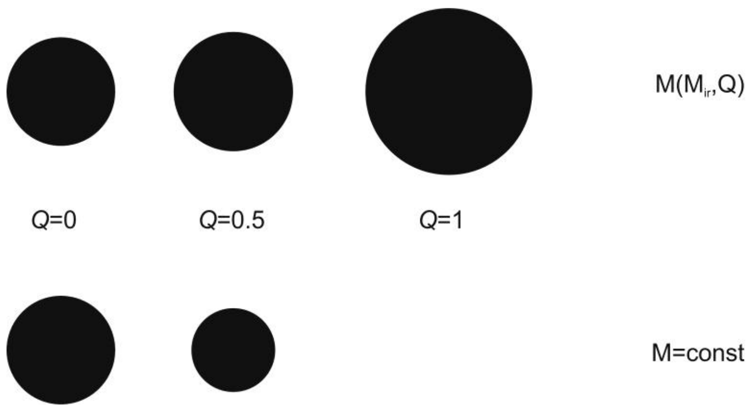

Fig. 5. Charge dependence of the shadow a charged collapsar in two approaches.

New curves for the dependence on Q of the radii of the orbits $r_{ph}^{(a)}$, $r_{ISCO}^{(a)}$ and the shadow $\lambda_c^{(b)}$ go up (Fig. 3), because with strengthening the gravity due to the increased energy of the electric field, the stable orbits become located at a greater distance from the source. In contrast to this the former curves for $r_{ph}^{(a)}$, $r_{ISCO}^{(a)}$, and $\lambda_c^{(a)}$ (under the condition $M = const$) go down (Fig. 4), which is absurd from a physical point of view. A more visual picture of the shadow behavior is shown in Fig. 5.

3. The Kerr metric in terms of irreducible mass and its consequences

3.1. The Kerr metric in terms of irreducible mass

A line element around a compact rotating source with axial symmetry, whose axis is directed along $\theta = 0$, takes the form:

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 + 2g_{03}dtd\varphi, \tag{22}$$

where t is world time and the metric has the form $g_{\mu\nu}(r,\theta)$. The standard form of the Kerr metric leads to the same line element in terms of Boyer-Lindquist coordinates:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \\ &-\left(r^2 + a^2 + \frac{2Mra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\, d\varphi^2 + \frac{4Mra}{\rho^2}\sin^2\theta d\varphi dt, \end{aligned} \tag{23}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \tag{24}$$

On the equatorial plane ($\theta = \pi/2$, $\rho^2 = r^2$) the Eq. (23) simplifies:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{r^2}{\Delta}dr^2 - \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)d\varphi^2 + \frac{4Ma}{r}d\varphi dt. \tag{25}$$

At the pole ($\theta = 0, \pi$; $\rho^2 = r^2 + a^2$) the Eq. (23) takes simplest form:

$$ds^2 = \frac{\Delta}{r^2 + a^2} dt^2 - \frac{r^2 + a^2}{\Delta} dr^2. \tag{26}$$

In the Kerr metric $Q = 0$ and the mass formula (3) takes the form:

$$M = \frac{M_{ir}}{\sqrt{1 - a^2 / 4M_{ir}^2}} \simeq M_{ir} + \frac{a^2}{8M_{ir}} + ... \tag{27}$$

Here M includes not only M_{ir}, but the mass equivalent of the rotational energy also and corresponding dependences on a of M and angular momentum $J = Ma$ are shown in Fig. 6-7.

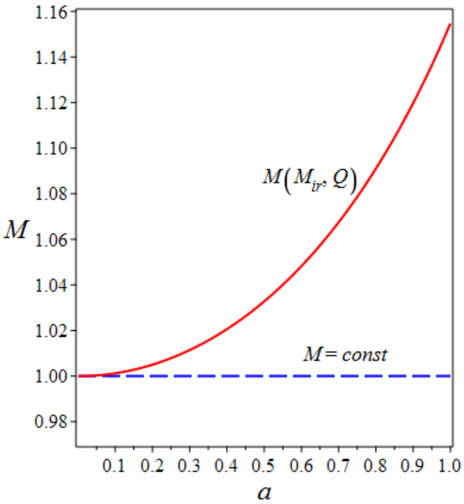

Fig. 6. Dependence on a of total mass M .

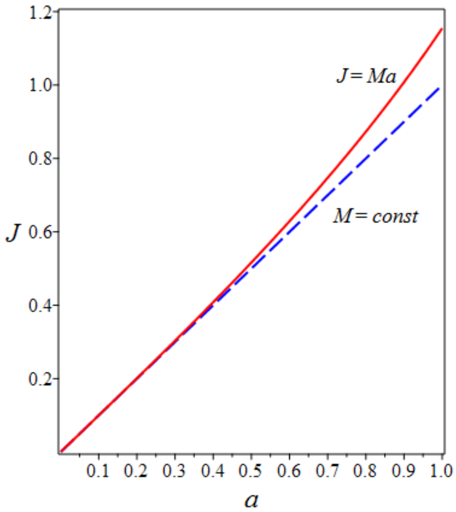

Fig. 7. Dependence on a of angular momentum.

Thus, the standard form of the Kerr metric (23) includes M as the main parameter, but this parameter depends on a , another parameter of the metric. The mass formula (27) allows to express M through independent parameters M_{ir}, a and this transforms the Kerr metric into the metric with independent parameters. Then the line element (23) takes the form:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2rM_{ir}}{\rho^2 \sqrt{1 - a^2 / 4M_{ir}^2}}\right) dt^2 - \frac{\rho^2}{\Delta_{ir}} dr^2 - \rho^2 d\theta^2 - \\ &- \left(r^2 + a^2 + \frac{2M_{ir} r a^2 \sin^2\theta}{\rho^2 \sqrt{1 - a^2 / 4M_{ir}^2}}\right) \sin^2\theta \, d\varphi^2 + \frac{4M_{ir} r a \sin^2\theta}{\rho^2 \sqrt{1 - a^2 / 4M_{ir}^2}} d\varphi dt, \end{aligned} \tag{28}$$

$$\Delta_{ir} = r^2 + a^2 - \frac{2rM_{ir}}{\sqrt{1 - a^2 / 4M_{ir}^2}}, \tag{29}$$

On the equatorial plane the Eq. (28) turns to:

$$ds^2 = \left(1 - \frac{2M_{ir}}{r\sqrt{1-a^2/4M_{ir}^2}}\right)dt^2 - \frac{r^2}{\Delta_{ir}}dr^2 - \\ -\left(r^2 + a^2 + \frac{2M_{ir}a^2}{r\sqrt{1-a^2/4M_{ir}^2}}\right)d\varphi^2 + \frac{4M_{ir}a}{r\sqrt{1-a^2/4M_{ir}^2}}d\varphi dt, \qquad (30)$$

and on the pole:

$$ds^2 = \frac{\Delta_{ir}}{r^2+a^2}dt^2 - \frac{r^2+a^2}{\Delta_{ir}}dr^2. \qquad (31)$$

At $r > 2M$ and $a < 2M_{ir}$ the metrics (29)-(31) do not have singularities.

Earlier, at studying the dependence on a of the observable effects in the gravitational field of a rotating source, it was erroneously assumed $M = const$ at changing a. Then this led to an absurd prediction that adding rotational energy weakens gravity and its effects [2-5,8-9].

In contrast with this, the metric with independent parameters in Eq. (29) shows that in fact the growth of a, leading to an increase M due to the contribution of the rotational energy, reduces g_{00} and thereby *strengthens* gravity. Aa the result, its observable effects, such as redshifts, average radii of orbits and shadows, *increase* with increasing a.

Thus, the transition to the Kerr metric with irreducible mass M_{ir} (28), performed by substituting (27) into (23), corrects previous errors and makes the predictions physically correct. The corresponding formulas and curves for the observable effects are given in Section 2.2.

3.2. Gravitational effects in the Kerr metric with irreducible mass

Equations and integrals of motion in the standard Kerr metric (23) with M together with formulas for the radii of special orbits are well known and are presented in [2-5,8-9]. To obtain the corresponding formulas for the Kerr metric with irreducible mass (28) it is enough to make the substitution (27) in all formulas for the observable effects. The new dependences on a that follow from such a replacement are presented below.

a. Time dilation and redshifts.

Gravitational time dilation at the point $(r,\theta,\varphi) = const$, i.e. slowing down of proper time τ with respect to t, is given by $\Delta\tau = \Delta t\, g_{00}^{1/2}(r,\theta)$. For two cases, ($a$) at the choice $M = const$ and (b) at $M(M_{ir,},Q)$, this gives:

$$\Delta\tau^{(a)} \simeq \Delta t\left(1 - \frac{2Mr}{\rho^2}\right)^{1/2}, \quad M = const. \qquad (32)$$

$$\Delta\tau^{(b)} \simeq \Delta t\left(1 - \frac{2M_{ir}r}{\rho^2\sqrt{1-a^2/4M_{ir}^2}}\right)^{1/2}, \qquad (33)$$

At the equator ($\theta = \pi/2$) these formulas are reduced to:

$$\Delta\tau^{(a)} \simeq \Delta t\left(1 - \frac{2M}{r}\right)^{1/2}, \quad M = const. \qquad (34)$$

$$\Delta\tau^{(b)} \simeq \Delta t\left(1-\frac{2M_{ir}}{r\sqrt{1-a^2/4M_{ir}^2}}\right)^{1/2}, \tag{35}$$

while on the pole ($\theta = 0$) :

$$\Delta\tau^{(a)} \simeq \Delta t\left(1-\frac{2M/r}{1+a^2/r^2}\right)^{1/2}, \quad \Delta\tau^{(b)} \simeq \Delta t\left(1-\frac{2M_{ir}/r}{(1+a^2/r^2)\sqrt{1-a^2/4M_{ir}^2}}\right)^{1/2}. \tag{36}$$

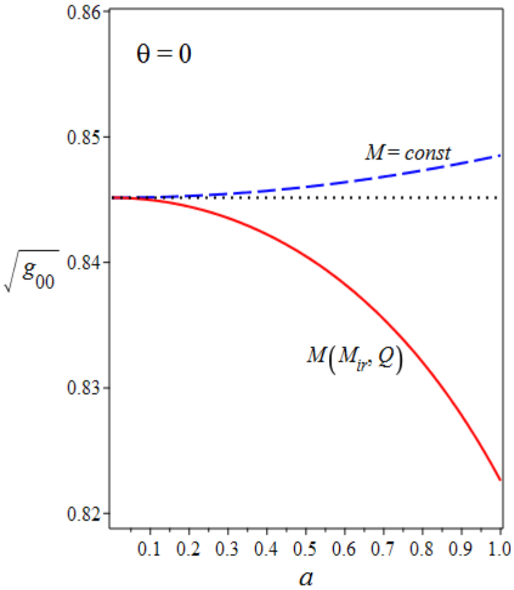

Fig. 8a. Dependence on a of time dilation at the poles, $r = 6M_{ir}$.

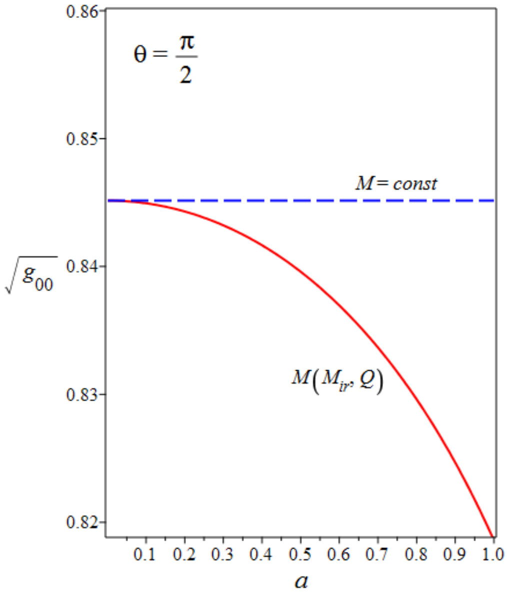

Fig. 8b. Dependence on a of time dilation at the equator, $r = 6M_{ir}$.

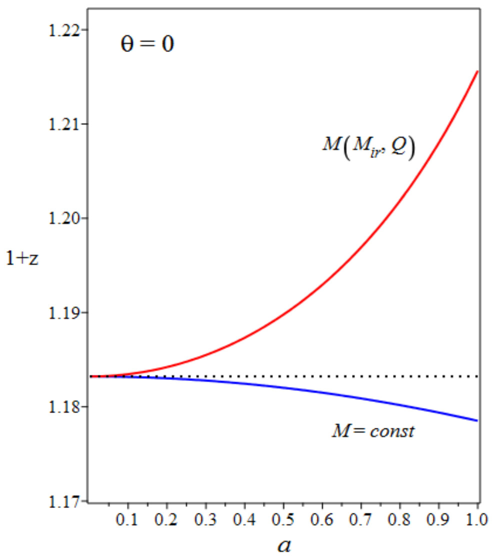

Fig. 9a. Dependence on a of redshift at the poles, $r = 6M_{ir}$.

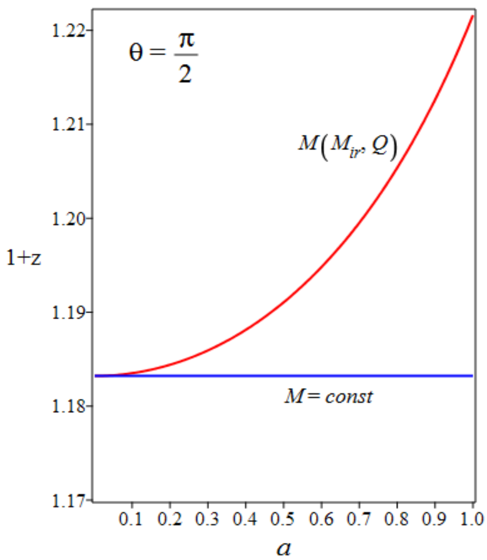

Fig. 9b. Dependence on a of redshift at the equator, $r = 6M_{ir}$.

Fig. 8 shows the difference in dependences on a in these two cases.

The dependence on a of the redshift factor $1+z=\Delta t/\Delta\tau=g_{00}^{-1/2}$ is inverse to the time dilation dependence and is presented on Fig. 9.

b. Radii of photon's circular orbits

The radii of the photon's circular orbit $r_{ph\pm}$ in the two directions of rotation on the orbit are given by:

$$r_{ph\mp}^{(a)}=2M\times$$

$$\times\left\{1+\cos\left[\frac{2}{3}\arccos\left(\mp\frac{a}{M}\right)\right]\right\}, \quad (37)$$

$$r_{ph\mp}^{(b)}=\frac{2M_{ir}(1+\cos F)}{\sqrt{1-a^2/4M_{ir}^2}}, \quad (38)$$

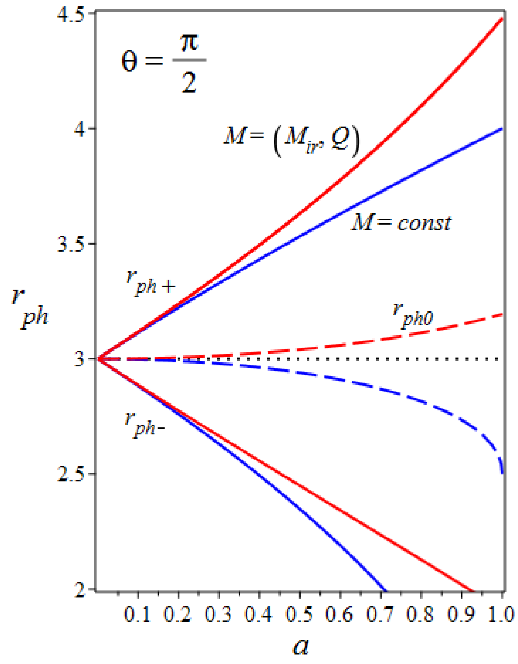

Fig. 10. Radii of photon orbits $r_{ph\pm}$ and r_{ph0}.

where

$$F=\frac{2}{3}\arccos\left(\mp\frac{a}{M_{ir}}\sqrt{1-\frac{a^2}{4M_{ir}^2}}\right).$$

Corresponding plots of the dependence on a of $r_{ph\pm}$ and the mean radius $r_{ph0}=(r_{ph+}+r_{ph-})/2$ (where the frame dragging effect is compensated), are presented in Fig. 10. The plots show that at $M=const$ the mean radius r_{ph0} decreases with increasing a, while in the case $M(M_{ir},Q)$, it increases with increasing a.

c. Radii of the marginally bound orbits.

Formulas for the radius of the marginally bound orbit $r_{mb\pm}$ with M and M_{ir} are:

$$r_{mb\mp}^{(a)}=2M\left(1+\sqrt{1\mp\frac{a}{M}}\right)\mp a, \quad (39)$$

$$r_{mb\mp}^{(b)}=\frac{2M_{ir}}{\sqrt{1-a^2/4M_{ir}^2}}\left(1+\sqrt{1\mp\frac{a}{M_{ir}}\sqrt{1-a^2/4M_{ir}^2}}\right)\mp a, \quad (40)$$

The plots for $r_{mb\pm}$ and $r_{mb0}=(r_{mb-}+r_{mb+})/2$ are shown in Fig. 11 .

d. Radii of the innermost stable circular orbits of particles.

The formula with M for the radii of the innermost stable circular orbits $r_{ISCO\pm}$ is:

$$r_{ISCO\mp}^{(a)}=M\{3+Z_2\mp\left[(3-Z_1)(3+Z_1+2Z_2)\right]^{1/2}\}, \quad (41)$$

$$Z_1 = 1 + (1-b^2)^{1/3}[(1+b)^{1/3} + (1-b)^{1/3}], \quad Z_2 = (3b^2 + Z_1^2)^{1/2}. \tag{42}$$

Here $b^{(a)} = a/M$ at $M = const$ and at expressing M in terms of M_{ir}, the Eq. (41) gives:

$$r^{(b)}_{ISCO\mp} = \frac{M_{ir}}{\sqrt{1-a^2/4M_{ir}^2}}\left\{3 + Z_2 \mp [(3-Z_1)(3+Z_1+2Z_2)]^{1/2}\right\},$$
$$b^{(b)} = \frac{a}{M_{ir}}\sqrt{1-a^2/4M_{ir}^2}, \tag{43}$$

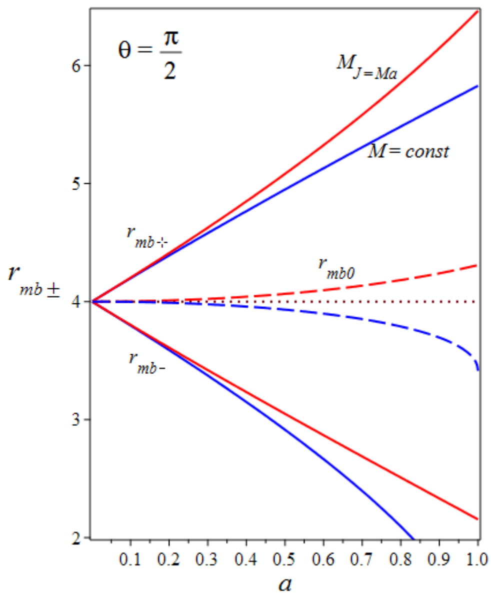

Fig. 11. Limit radii of the orbit of a bound state $r_{mb\pm}$ and their mean r_{mb0} in the Kerr metric.

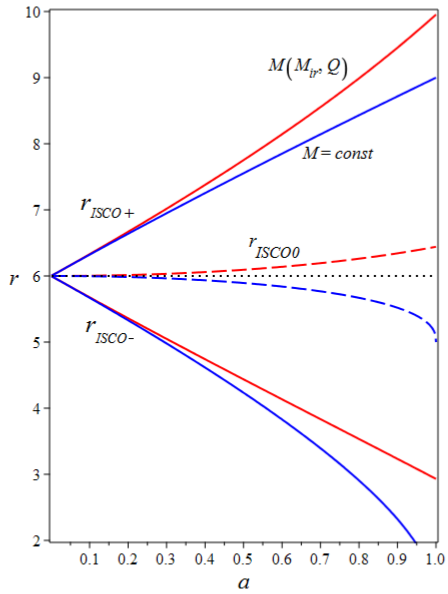

Fig. 12. Radii of the innermost stable circular orbits of particles $r_{ISCO\pm}$ and their mean r_{ISCO0} in the standard Kerr metric.

Plots for $r_{ISCO\pm}$ and $r_{ISCO0} = (r_{ISCO+} + r_{ISCO-})/2$ in both cases are shown in Fig. 12 .

e. The shadow of a rotating collapsar.

In the Kerr metric, the light-like equations of geodesics are separable and their solution is simplified by the presence of four constants of motion - the rest mass of a photon (equal to zero), energy E, angular momentum L and Carter's constant K. The coordinates (x, y) of the edge of the shadow on the image plane, visible from afar on the equatorial plane, are related to these constants as $x \simeq -L/E$ and $y = \pm\sqrt{K}/E$. For a spherical orbit of a photon of radius r, they satisfy the relations:

$$x = \frac{r^3 + a^2 r + (a^2 - 3r^2)M}{a(r-M)}, \quad y = \pm\sqrt{\frac{3r^2 + a^2 - x^2}{1 - a^2/r^2}}. \tag{44}$$

The first relation (44) is the equation for r:

$$r^3 - 3Mr^2 + a(a-x)r + Ma(a+x) = 0, \tag{45}$$

which can be solved by Vieta's trigonometric method [16]. By using notations

$$A(x) \equiv 1 - \frac{a(a-x)}{3M^2}, \quad B(x) \equiv \frac{1}{|A(x)|^{3/2}}\left(1 - \frac{a^2}{M^2}\right), \tag{46}$$

the solution can be written as $r(x) = M \cdot \tilde{r}(x)$, where

$$\tilde{r}(x) = 1 + 2\sqrt{A}\cos\left(\frac{1}{3}\arccos B\right), \qquad A > 0,\ B \le 1, \tag{47}$$

$$\tilde{r}(x) = 1 + 2\sqrt{A}\cosh\left(\frac{1}{3}\ln\left(\sqrt{B^2 - 1} + B\right)\right), \qquad A \ge 0,\ B > 1, \tag{48}$$

$$\tilde{r}(x) = 1 - 2\sqrt{|A|}\sinh\left(\frac{1}{3}\ln\left(\sqrt{B^2 + 1} - B\right)\right), \qquad A < 0. \tag{49}$$

To obtain the contour of the shadow for a given x, we calculate $r(x)$ from (47)-(49), then M express through a according to the mass formula (27) and, substituting the result into the second relation (44), we find $y(x)$. The plots of the contour of the collapsar's shadow are shown in Fig. 13. For comparison, the contours of the shadow are also shown for the choice $M = const$ (Fig. 14). With growth a, the shadow area in the first case decreases, and in the second case it increases.

Thus, in the Kerr metric with irreducible mass, the growth of a, which means an increase in the rotational energy and strengthening gravity around the source, leads to an increase in redshifts, radii of special orbits, and shadow area, as it should be from general physical considerations.

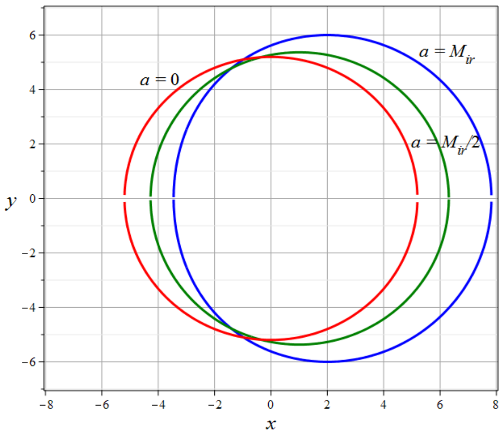

Fig. 13. Contour of the shadow of a rotating collapsar in the Kerr metric with irreducible mass, $a = (0;\ 0.5;\ 1)M_{ir}$.

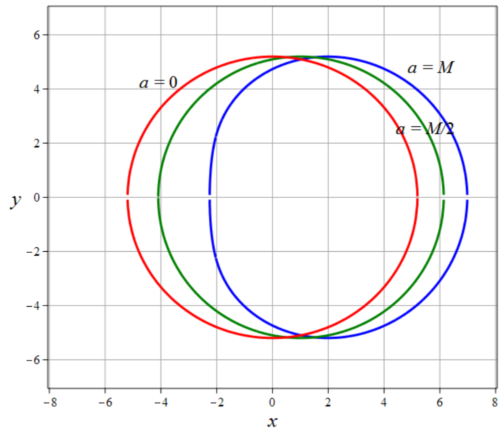

Fig. 14. Contour of the shadow of a rotating collapsar in the standard Kerr metric with $M = const$, $a = (0;\ 0.5;\ 1)M_{ir}$.

4. The Kerr-Newman metric with irreducible mass and its consequences

4.1. The Kerr-Newman metric with irreducible mass

The KN metric around a compact source with a rotation parameter a and a charge Q follows from the standard Kerr metric by replacing $2Mr \to 2Mr - Q^2$. Then, the line element (23) takes the form [1-5]:

$$ds^2 = \left(1 - \frac{2Mr - Q^2}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta^{(Q)}}dr^2 - \rho^2 d\theta^2 + \frac{2a(2Mr - Q^2)}{\rho^2}\sin^2\theta\, dt d\varphi - \left(r^2 + a^2 + \frac{a^2(2Mr - Q^2)}{\rho^2}\sin^2\theta\right)\sin^2\theta\, d\varphi^2, \quad (50)$$

$$\Delta^{(Q)} = r^2 - 2Mr + a^2 + Q^2. \quad (51)$$

In the metric in Eq. (50), its main parameter, total mass M, depends on two other parameters a, Q and, therefore, M must be expressed through M_{ir}, a and Q according to the mass formula (3) (Fig. 15). After that, the KN metric takes the form:

$$ds^2 = \left(1 - \frac{2rM_{ir}}{\rho^2}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} + \frac{Q^2}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta_{ir}^{(Q)}}dr^2 - \rho^2 d\theta^2 - \left[r^2 + a^2 + \frac{a^2}{\rho^2}\left(2M_{ir}r\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} - Q^2\right)\sin^2\theta\right]\sin^2\theta\, d\varphi^2 + \frac{2a}{\rho^2}\left(2M_{ir}r\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} - Q^2\right)\sin^2\theta\, dt d\varphi, \quad (52)$$

$$\Delta_{ir}^{(Q)} = r^2 - 2rM_{ir}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} + a^2 + Q^2. \quad (53)$$

On the equatorial plane (52) goes into:

$$ds^2 = \left(1 - \frac{2M_{ir}}{r}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} + \frac{Q^2}{r^2}\right)dt^2 - \frac{r^2}{\Delta_{ir}^{(Q)}}dr^2 - (r^2 + a^2)d\varphi^2 - \frac{a^2}{r}\left(2M_{ir}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} - \frac{Q^2}{r}\right)d\varphi^2 + \frac{2a}{r}\left(2M_{ir}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} - \frac{Q^2}{r}\right)d\varphi dt, \quad (54)$$

and on the pole:

$$ds^2 = \frac{\Delta_{ir}^{(Q)}}{r^2 + a^2}dt^2 - \frac{r^2 + a^2}{\Delta_{ir}^{(Q)}}dr^2. \quad (55)$$

At $r > 2M$ and $a < 2M_{ir}$ the metrics (50)-(55) do not have singularities. In them, the growth of a and Q, leading to an increase in M due to the rotational energy, decreases g_{00} and therefore *strengthens* gravity. As a result, the observable effects in the field of a rotating charged source (redshifts, mean radii of orbits and shadow) *increase* with a and Q.

Previous predictions under the condition $M = const$ [2-5,8-9] should then be replaced with correct predictions taking into account the dependence M on M_{ir}, a and Q. As examples, predictions for the time dilation effect and redshifts will be presented below.

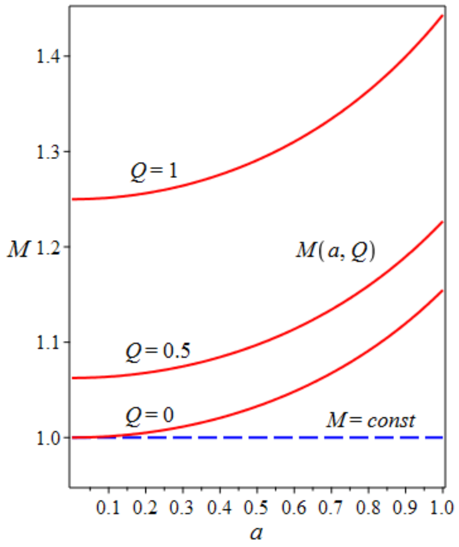

Fig. 15. Total mass of the source in the KN metric.

4.2. Proper time dilation and redshifts

The slowdown of the proper time interval $\Delta\tau = \Delta t\, g_{00}^{1/2}(r,\theta)$ between two events around the source with angular momentum and charge, according to (50), is given by the expressions, respectively: (a) at $M = const$ and (b) at using the mass formula (3):

$$\Delta\tau^{(a)} \simeq \Delta t\left(1 - \frac{2Mr}{\rho^2} + \frac{Q^2}{\rho^2}\right)^{1/2}, \tag{56}$$

$$\Delta\tau^{(b)} \simeq \Delta t\left(1 - \frac{2M_{ir}r}{\rho^2}\frac{1 + Q^2/4M_{ir}^2}{\sqrt{1 - a^2/4M_{ir}^2}} + \frac{Q^2}{\rho^2}\right)^{1/2}, \tag{57}$$

The Eq. (57), written as:

$$\Delta\tau^{(b)} \simeq \Delta t\left\{1 - \frac{2M_{ir}r}{\rho^2\sqrt{1 - a^2/4M_{ir}^2}}\left[1 + \frac{Q^2}{4M_{ir}^2}\left(1 - \frac{2M_{ir}}{r}\sqrt{1 - a^2/4M_{ir}^2}\right)\right]\right\}^{1/2}, \tag{58}$$

clearly shows that the growth of a and Q increases the time dilation effect.

At the equator ($\theta = \pi/2$), these formulas take the form:

$$\Delta\tau^{(a)} \simeq \Delta t\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}, \tag{59}$$

$$\Delta\tau^{(b)} \simeq \Delta t\left\{1 - \frac{2M_{ir}}{r\sqrt{1 - a^2/4M_{ir}^2}}\left[1 + \frac{Q^2}{4M_{ir}^2}\left(1 - \frac{2M_{ir}}{r}\sqrt{1 - a^2/4M_{ir}^2}\right)\right]\right\}^{1/2}, \tag{60}$$

and on the pole ($\theta = 0$):

$$\Delta\tau^{(a)} \simeq \Delta t\left(1 - \frac{2M}{r(1 + a^2/r^2)} + \frac{Q^2}{r^2 + a^2}\right)^{1/2}, \tag{61}$$

$$\Delta\tau^{(b)} \simeq \Delta t\left\{1 - \frac{2M_{ir}(1 + a^2/r^2)^{-1}}{r\sqrt{1 - a^2/4M_{ir}^2}}\left[1 + \frac{Q^2}{4M_{ir}^2}\left(1 - \frac{2M_{ir}}{r}\right)\sqrt{1 - a^2/4M_{ir}^2}\right]\right\}^{1/2} \tag{62}$$

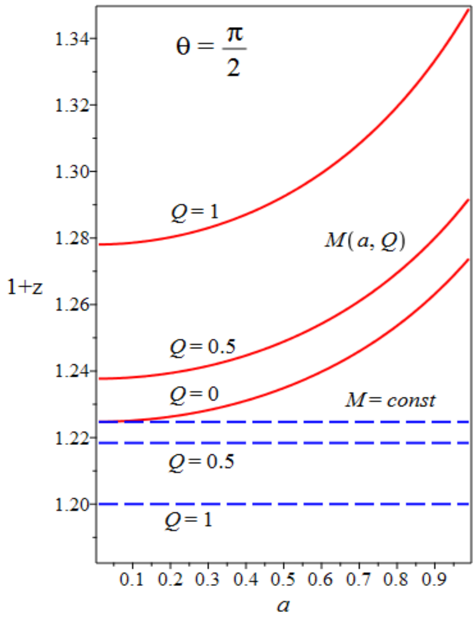

Fig. 16a. Dependence on a of redshift at the equator, $r = 3M_{ir}$.

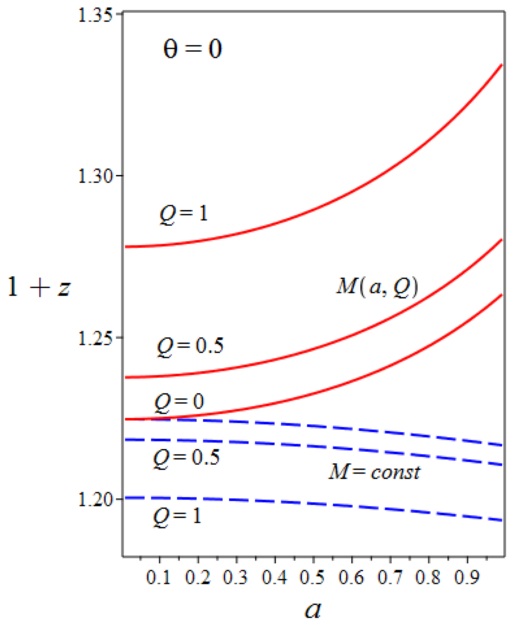

Fig. 16b. Dependence on a of redshift at the equator, $r = 3M_{ir}$.

The corresponding plots of dependence on a and Q of the redshift factor $1+z = g_{00}^{-1/2}$ following from (59)-(62) are shown in Fig. 16.

Thus, an increase in a and Q in the KN metric with irreducible mass decreases $g_{00}^{1/2}$, i.e., increases gravity, increasing the degree of slowing down of proper time and the magnitude of redshifts. As in the case of the Kerr metric, other effects of gravity are also amplified.

5. Conclusion

The standard form of the KN metric outside the charge and rotating sources includes three parameters and one of them, the total mass at infinity M, depends on other two, a and Q. In the paper, the metrics is reduced to the form with independent parameters, in which M is expressed in terms of the reduced mass M_{ir}, defined as the remainder of the total mass at $a = Q = 0$.

This made it possible to eliminate previous predictions about the strange behavior of gravity and its effects at changing a and Q. These predictions were under the assumption $M = const$ at changing a and Q, and consisted of statements that the growth of a and Q, leading to increasing the energies of rotation and the electric field, weakens gravity and its effects outside the source [2-9] .

It is shown that from the RN, Kerr and KN metrics with independent parameters M_{ir}, a and Q the opposite predictions follow that the growth of a and Q, adding positive energies of rotation and electric field, strengthens gravity and increases the magnitude of its effects. These consequences are physically correct and in the future (for physically realizable values of the parameters) will be confirmed by observations in the same way as it has always been with physically correct predictions of general relativity.

In conclusion, it should be noted that, from a physical point of view, it is also necessary to take into account the negative binding energy of the electric potential that holds ions on the surface of a source with radius r_b. This binding energy sets upper limits on the source's parameters $a < a_{\max}$ and $Q < Q_{\max}$, above which the ions leave the source due to centrifugal or Coulomb repulsion forces. In the paper, it has been assumed that this binding energy is sufficient to hold the ions, and that this energy is already included in M_0. Since for most states of matter in the sources $a_{\max}^2 / r_b \ll M_0$ and $Q_{\max}^2 / r_b \ll M_0$, the plots presented in the paper are physically justified for the same small values of a and Q. The given plots for large values of a and Q have purely theoretical nature and are of interest mainly at compared with similar plots without taking into account the dependence M on a and Q.

A more detailed description of the consequences of metrics with independent parameters and their further applications is given in the book [12].

References

1. Misner C., Thorne K., Wheeler J. (1973). Gravitation. Macmillan.
2. Frolov V., Zelnikov A. (2011) Intr. to BH Physics. Oxf. U. P.
3. Bambi C. (2017) *Black Holes: A Lab. for Testing Strong Gravity*. Springer.
4. Grenzebach A. (2017) The Shadow of BHs. Springer.
5. Ferrari V., Gualtieri L., Pani P. (2021) *Gen. Rel. and its Applications*. Taylor & Fr. G.
6. Jeffery G. (1921) *Proc. R. Soc. Lond*. **A99** (697) 123.
7. Pugliese D., Quevedo H., Ruffini R. (2011) *Phys. Rev*. **D84**, 044030.
8. Bardeen J., Press W., Teukolsky S. (1972) *Astrophys. J.* **178**, 347.
9. Pugliese D., Quevedo H., Ruffini R. (2013) *Phys. Rev*. **D88**, 024042.
10. Zakir Z. (2022) *Quant. and Grav. Phys*, **3**:016-7694, **3**:017-7694,
11. Zakir Z. (2022) *Quant. and Grav. Phys*., **3**:018-7715.
12. Zakir Z. (2023) *Metrics with Indep. Param. outside Charged and Rotat. Sources.* CTPA, T.
13. Florides P. (1960) *PhD Thesis, U. London*; (1962) *Math. Proc. Cambr. Phil. Soc*. **58**(1) 110; (1977) *Nuovo Cim*., **A42**(3), 343.
14. Christodoulou D. (1970) *Phys. Rev. Lett*., **25**, 1596.
15. Christodoulou D., Ruffini R. (1971) *Phys. Rev*., **D4**, 3552.
16. Cunha P. V., Herdeiro C. A. (2018) *Gen. Rel. & Grav*., **50**(4), 1.